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Research Article

Enumerating Structures: Applications of the Hyperoctahedral Group in Combinatorial Species

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Abstract

Combinatorial species provide a framework for counting and classifying for counting and classifying combinatorial structures. A species assigns a set of structures to each finite set, respecting the notion of isomorphism. This approach facilitates the enumeration of various combinatorial objects. The hyperoctahedral group, also known as the signed permutation group, consists of permutations of a set of signed elements. This group plays a crucial role in combinatorial algebra, particularly in the enumeration of certain structures, such as various types of trees and graphs. Generating series, both ordinary and exponential, are powerful tools in combinatorial enumeration. Combining these concepts allows for deeper insights into the relationships between structures and their enumerative properties, paving the way for advanced combinatorial theory and applications in various mathematical fields.

Introduction

In combinatorial mathematics, the study of structures and their classifications is a vibrant area that often employs the concept of combinatorial species. A combinatorial species [1-3] provides a framework for counting and analyzing different structures (like graphs, trees, or permutations) by focusing on their combinatorial properties rather than their specific representations. This approach facilitates the comparison and enumeration of various configurations through the use of generating series [4,5].

An essential tool in this field is the hyperoctrahedral group, which arises in the study of symmetries of higher-dimensional geometric objects. Specifically, the hyperoctahedral group describes the symmetries of an n -dimensional cube [6], capturing the essence of permutations and reflections of its vertices. This group plays a crucial role in combinatorial enumeration and can be linked to various combinatorial species through its action.

To systematically analyze these species, mathematicians use ordinary generating series and exponential generating series. Ordinary generating series are particularly useful for counting sequences of combinatorial structures, while exponential generating series are tailored for structures where order matters, such as labeled objects.

The study of combinatorial species and the hyperoctahedral group is vital in understanding symmetrical structures and their properties in various mathematical contexts. Combinatorial species provide a framework for counting and categorizing combinatorial objects based on their structural characteristics, allowing for a deeper insight into their relationships and transformations.

This article, section 2 gives the theoretical foundations of combinatorial species and the hyperoctrahedral group B_n as a wreath product. The section 3 is devoted to the enumeration techniques based on the hyperoctrahedral group B_n.

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Theoretical foundations

Combinatorial species: We consider the category B whose objects are the finite sets whose morphisms are the bijections, and the category εns of the finite sets whose morphisms are the functions.

Definition 2.1 *[7] A species of structures is a functor* $F : B \to \mathcal{E}ns$.

This means that to any finite set U, we associate a finite set noted $F[U]$ whose elements are called the F-structures on U. To any bijection $\beta : U \tilde{\to} V$, we associate a function $F[\beta] : F[U] \to F[V]$, which we call transport morphism along β from the F -structures on U to the F-structures of V. Moreover, we ask for the functoriality properties: For $U[r]^{\beta} V[r]^{\alpha} W$, we have $F[\alpha \circ \beta] = F[\alpha] \circ F[\beta]$ and for $1_U : U \to U$, the identity of U, $F[1_U] = 1_{F[U]}$.

It follows that for any bijection β the transport morphism F[β] is a bijection. Indeed, for $U[r]^{\beta} V[r]^{\beta^{-1}} U$, we have

$$F[\beta^{-1}] \circ F[\beta] = F[\beta^{-1} \circ \beta] = F[1_U] = 1_{F[U]}$$

Similarly, we have $F[\beta] \circ F[\beta^{-1}] = 1_{F[V]}$.

So, $F[\beta]$ admits as inverse $F[\beta^{-1}]$.

Isomorphisms of species and isomorphism of F-structures:

Definition 2.2 *[8] The species F and G are said to be isomorphic (and we write* $F \simeq G$ *) if there exists a natural isomorphism* ϑ *between the functors F and G.*

This means that to any finite set U, we associate a bijection $\vartheta_U : F[U] \to G[U]$ such that for any bijection $\beta : U \tilde{\to} V$, we have $G[\beta] \circ \vartheta_U = \vartheta_V \circ F[\beta]$. The following diagram commutes

$$F[U][r]^{F[\beta]}[d]_{\vartheta_U} \; F[V][d]^{\vartheta_V} \; G[U][r]^{G[\beta]} G[V]$$

Definition 2.3 *Two F-structures* t_1 *on U and* t_2 *on V are said to be isomorphic, if there exist* $f : U \tilde{\to} V$*, bijection, such that* $F[f](t_1) = t_2$*. We then write* $t_1 \cong t_2$*. The relation (being isomorphic) is an equivalent relation on* $T[U]$*, and we denote the set of equivalence classes (or isomorphy classes or types of F-structures on U by* $F[U]/_{\cong}$.

Proposition 2.1 Let $\mathfrak{S}_U$ and $\mathfrak{S}_{F[U]}$ be the groups of permutations of U and $F[U]$ respectively. Then the map defined by the following is a homomorphism of groups

$h : \mathfrak{S}_U$	$\to$	$\mathfrak{S}_{F[U]}$
σ	$\mapsto$	$F[\sigma]$

Proof 2.1 *We have:* $\forall \sigma, \tau \in \mathfrak{S}_U$;

$$h(\sigma \circ \tau) = F[\sigma \circ \tau] = F[\sigma] \circ F[\tau] = h(\sigma) \circ h(\tau).$$

Remark 2.1 *This amounts to saying that we have an action of* $\mathfrak{S}_U$ *on* $F[U]$*. We note, that for* $t \in F[U]$, $aut(t) = Stab(t) = \{\sigma \in \mathfrak{S}_U \mid F[\sigma](t) = t\}$ *we call the group of automorphisms of t. We will write* $\sigma \cdot t$ *instead* $F[\sigma](t)$.

Remark 2.2 *For any species F, there exists a (called associated) species* $\tilde{F}$ *defined by:*

$$\tilde{F}[U] = \{(t,\sigma) \mid t \in F[U], \sigma \in \mathfrak{S}_U \text{ and } \sigma \cdot t = t\}$$

And for any bijection $f : U \tilde{\to} V$

$\tilde{F}[f]\;\; \tilde{F}[U]$	$\to$	$\tilde{F}[V]$
(t,σ)	$\mapsto$	$(F[f](t), f \circ \sigma \circ f^{-1})$

Remark 2.3 *When* $aut(t) = \{id_U\}$*, we say that t is an asymmetric (or rigid or flat) F-structure on U.*

Definition 2.4 *The flat part of a species F is the subspecies* $\overline{F} \subseteq F$ *defined by* $\overline{F}[U] = \{t \in F[U] \mid t \text{ is asymmetric}\}$ *where the transport morphisms along the bijections are obtained by restriction.*

Remark 2.4 *We have* $\overline{F} \subseteq F \subseteq \tilde{F}$ *where we identify* t *with (t,id). In a certain sense* $\overline{F}$ *measures the asymmetry and* $\tilde{F}$ *the symmetry of the F-structures.*

Hyperoctahedral group: The symmetric group $\mathfrak{S}_n$ can be considered as a matrix group where $n \times n$-matrices are permutation matrices (only one 1 per row and column, 0elsewhere) [9,10].

Let $\sigma \in \mathfrak{S}_n$, $\forall i \in [n]$; $\sigma(i) = j$, $j \in [n]$.

$\sigma(i) = j \Rightarrow a_{ji} = a_{\sigma(i)i} = 1$ at the i^{th} column. This practice generalizes for the hyperoctahedral group, B_n, with the difference that the 1 of each row and column can be replaced by -1 (matrices of signed permutations).

Take the following matrix as an example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

that can be noted $(1,1,-1;(1)(23)) = (f,\pi)$ where $\pi \in \mathfrak{S}_3$, $f : [3] \to \{1,-1\}$ and $f(i)$ denotes the non-zero element of the i^{th} row of the matrix. We can generalize this notation by the following definition.

Definition 2.5 *(Hyperoctahedral group as a wreath product)*

Let G be a finite group, H a subgroup of $\mathfrak{S}_n$. Let's pose:

$$G \wr H = \{(f,\pi) \mid f : [n] \to G; \pi \in H\}.$$

$G \wr H$ is a group with the following composition

$$(f,\pi)(f',\pi') = (ff'_{\pi}, \pi\pi')$$

where f'_{π} denotes $f' \circ \pi^{-1}$ and $(ff'')(i) = f(i)f''(i)$, $i \in [n]$. The identity element is (e, id_H) where $e(i) = id_G$, $i \in [n]$. The inverse

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$(f,\pi)^{-1} = (f^{-1}_{\pi^{-1}};\pi^{-1})$.

B_n is isomorphic to $\mathfrak{S}_2 \wr \mathfrak{S}_n$ [3], when the notation is abused by replacing in the permutation matrix $\{id_{\mathfrak{S}_2}\ and (12)\}$ by 1 and -1 respectively. In other words, we have: $B_n = \mathbb{Z}_2 \wr \mathfrak{S}_n$.

The hyperoctrahedral group B_n has subgroups:

- $G^* = \{(f,id_H) | f \in G^n\}$;
- $H' = \{(e_{G^n},\pi) | \pi \in H\}$;
- $Diag(G^*) = \{(f,id_H) | f(i) = g, i \in [n], g \in G\}$.

In general, if we have a finite group G and H a subgroup of $\mathfrak{S}_n$, the wreath product $G \wr H$ is the right semi-direct product of G^n by H with the following operation:

$$(g'_1,\dots,g'_n,s')(g_1,\dots,g_n,s) = (g'_{s(1)}g_1,\dots,g'_{s(n)}g_n,s's).$$

In particular, it is easy to verify that $(1_G,\dots,1_{G'},1_H)$ is the unit element of $G \wr H$ and the inverse of $(g_1,\dots,g_n,s)$ is $(g^{-1}_{s^{-1}(1)},\dots,g^{-1}_{s^{-1}(n)},s^{-1})$.

Given the coxeter group of type B_n ; $B_n = \mathbb{Z}_2 \wr \mathfrak{S}_n$ is the wreath product of $\mathbb{Z}_2$ with the symmetric group $\mathfrak{S}_n$. So the following sequence is exact and short.

$$0[r]\,\mathbb{Z}_2^n[r]^{i}\,B_n[r]^{\pi}\,\mathfrak{S}_n[r]\,0$$

Moreover, the section s verifies $\pi \circ s = id_{\mathfrak{S}_n}$.

Generating series of a species of structures on B_n

The radius of convergence: In mathematics, the radius of convergence refers to the interval within which a power series converges to a function. For a power series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n, \tag{1}$$

Where a_n are the coefficients and c is the center of the series, the radius of convergence R can be determined using the formula:

$$\frac{1}{R} = \lim_{n\to\infty} sup\, |a_n|^{\frac{1}{n}} \tag{2}$$

This means that the series converges for all x such that $|x-c| < R$ and diverges for $|x-c| > R$. At the endpoints $|x-c| = R$, the behavior of the series must be checked individually.

Generating series on B_n:

Definition 3.1 *The generating series of a species of structures F on B_n is the formal power series*

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{2^n \times n!}, \tag{3}$$

where $f_n = |F[n]| =$ the number of F-structures on a set of n elements (labelled structures). Note that this series is a hyperoctahedral exponential type in the indeterminate x in sense that $2^n \times n!$ appears in the denominator of the term of degree n.

The series $F(x)$ is also called the hyperoctahedral exponential generating series of the species F. The following notation is used to designate the coefficients of formal power series. For a hyperoctahedral ordinary formal power series

$$G(x) = \sum_{n\geq 0} g_n x^n, \tag{4}$$

we set

$$[x^n]G(x) = g_n. \tag{5}$$

For a formal power series of hyperoctahedral exponential type, we then have

$$(2^n \times n!)[x^n]F(x) = f_n. \tag{6}$$

Relationship between hyperoctahedral ordinary and hyperoctahedral exponential generating function:

Lemma 3.1 *Let* $(g_n)_{n\geq 0}$ *, let*

$$G(x) = \sum_{n\geq 0} g_n x^n \tag{7}$$

be its hyperoctahedral ordinary generating function and

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{2^n \times n!} \tag{8}$$

its hyperoctahedral exponential generating function. Then $F(x)$ has an infinite radius of convergence.

Proof 3.1 *Let $R > 0$ be arbitrary. Let us show that for all $|x| \leq R$, the series*

$$\sum_{n=0}^{\infty} |f_n \frac{x^n}{2^n \times n!}| \tag{9}$$

converges. By hypothesis, there exist $c > 0$ and $d > 0$ such that $|f_n| \leq c \cdot d^n$ for all n. We have for all $|x| \leq R$:

$$|f_n \frac{x^n}{2^n \times n!}| \leq \frac{c\cdot(d\cdot R)^n}{2^n \times n!}, \quad \forall n \geq 0 \tag{10}$$

and so

$$\sum_{n=0}^{\infty} |f_n \frac{x^n}{2^n \times n!}| \leq c \sum_{n=0}^{\infty} \frac{(d\cdot R)^n}{2^n \times n!} = c\exp(d\cdot R). \tag{11}$$

Proposition 3.1 *Let $(g_n)_{n\geq 0}$ be a sequence whose hyperoctahedral ordinary generating function in $G(x)$ and whose hyperoctahedral exponential generating function is $F(x)$. Let $R > 0$ the radius of convergence of $G(x)$. Then, we have for all $|x| < R$:*

$$G(x) = \int_0^{\infty} e^{-t} F(xt)dt \tag{12}$$

and the integral converges uniformly on the disk $\{x : |x| < r\}$ where $0 < r < R$.

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Proof 3.2 *Fix $0<r<R$. We choose R' such that $0<r<R'<R$ and we set $\rho=\frac{r}{R'}<1$. By hypothesis, there exists $c>0$ such that $|f_n|R'^n\leq c$ for all n since the series $\sum_n f_n R'^n$ converges and therefore*

$$|f_n x^n|\leq|f_n|r^n\leq c\cdot\rho^n \quad \forall n. \tag{13}$$

Then we have for all integer $N>0$ and for all $|x|\leq r$:

$$|\sum_{n=0}^{N} f_n x^n \frac{t^n e^{-t}}{2^n\times n!}|\leq \sum_{n=0}^{N} c\rho^n \frac{t^n e^{-t}}{2^n\times n!}=ce^{-t}\sum_{n=0}^{N}\frac{(c\rho)^n}{2^n\times n!}\leq ce^{-(1-\rho)t} \tag{14}$$

which is integrable over $[0,+\infty[$. By the Dominated Convergence Theorem, we get:

$$\int_0^{\infty} e^{-t}F(xt)dt=\sum_{n=0}^{\infty} f_n x^n \int_0^{\infty}\frac{t^n e^{-t}}{2^n\times n!}dt=\sum_{n=0}^{\infty} g_n x^n \tag{15}$$

since $\int_0^{\infty} t^n e^{-t}dt=2^n\times n!$ for all n.

Finally, we also have

$$|\sum_{n=0}^{\infty} f_n x^n \frac{t^n e^{-t}}{2^n\times n!}|\leq ce^{-(1-\rho)t} \quad \forall |x|\leq r \tag{16}$$

and then

$$\sup_{|x|\leq r}|\int_0^{b} e^{-t}F(xt)dt-G(x)|=\sup_{|x|\leq r}|\int_b^{\infty} e^{-t}F(xt)dt|$$

$$\leq\int_b^{\infty} ce^{-(1-\rho)t}dt$$

$$=\frac{ce^{-(1-\rho)b}}{1-\rho}$$

which tends to 0 when $b\to\infty$.

Conclusion

In conclusion, the study of the hyperoctahedral group and its applications in combinatorial species reveals profound connections between algebra and combinatorics. By enumerating structures through the lens of the hyperoctahedral group, we uncover rich combinatorial objects and relationships that enhance our understanding of symmetry and structure. This interplay not only provides a robust framework for classifying and counting various configurations but also opens avenues for further exploration in both mathematical theory and practical applications.

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