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Review Article

A Poisson "Half-Summation" Formula

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Abstract

A generalization of Poisson's summation formula is derived – in a non-rigorous way – allowing evaluation of sums from 1 (or any finite integer) ∞ instead of the usual range -∞+∞. This is achieved in two ways, either by introducing a converging factor in a geometric series of exponential functions and letting it approach zero in a controlled way or by applying a Hilbert transform to the series. Several examples illustrate its usefulness in the evaluation of series and specific applications.

1. Introduction

The Poisson summation formula (see, e.g., Ref. [1], ch. 5.4) gives a relation between the continuous Fourier transform and the Fourier series coefficients (of the periodic summation) of a function $F(x)$

$$\sum_{m=-\infty}^{\infty} F\left(\lambda m\right) = \frac{1}{\lambda} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} dx\, F(x) \exp\left(-2\pi i n \frac{x}{\lambda}\right). \quad (1.1)$$

Here $\lambda > 0$ is a free parameter. Examples 41 – 43 in Ref. [1] demonstrate how it can be used to accelerate the convergence of an infinite sum or give closed expressions of it. It is also widely employed in many applications in physics, one example being given by the world line variational approach in Ref. [2].

In the following Poisson's formula will be written as

$$\sum_{m=-\infty}^{\infty} \delta\left(x-\lambda m\right) = \frac{1}{\lambda} \sum_{n=-\infty}^{\infty} \exp\left(-2\pi i n \frac{x}{\lambda}\right) \quad (1.2)$$

With Dirac's delta distribution as a generalized function. The left-hand side of Eq. (1.2) is also called a "Dirac comb" [3]. Due to the well-known relation (see, e.g. Ref. [4], eq. (A.18))

$$\delta\left(g(x)\right) = \sum_m \frac{1}{|\, g'(x_m)\,|} \delta\left(x - x_m\right) \quad \text{with} \quad g(x_m) = 0, \quad g'(x_m) \neq 0 \quad (1.3)$$

It may be written as

$$\sum_{m=-\infty}^{\infty} \delta\left(x-\lambda m\right) = \frac{\pi}{\lambda} \delta\left[\tan\left(\frac{\pi x}{\lambda}\right)\right]. \quad (1.4)$$

Eq. (1.2) is particularly useful if multiplied by a test function with finite support and integrated over because then the left-hand side reduces to a few terms in which the δ-"functions" have become active whereas the right-hand side still has the infinite sum over Fourier components. In this way, Poisson's summation formula can be used to evaluate infinite sums. However, some care has to be exercised in this procedure: if the -"functions" have their singularity right on the edge of the integration range they only contribute with a factor 1/2 as they are symmetrical across the borderline. This is seen in Example 1 of Section 4 and more generally for

$$F(x) =: \Theta(x) f(x) \quad (1.5)$$

Which $\lambda = 1$ leads to the usual form of Poisson's summation formula

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$$\frac{1}{2}f(0)+\sum_{m=1}^{\infty}f(m)=\int_0^{\infty}dx\,f(x)+2\sum_{n=1}^{\infty}\int_{-\infty}^{+\infty}dx\,f(x)\cos\left(2\pi nx\right) \tag{1.6}$$

(eq. 1.8.15 in Ref. [5]).

In general, however, I will proceed with the usual sloppiness of theoretical physicists who want to get reasonable results – even with questionable methods[1]. Here this is exemplified by the handling of distributions or in the neglect of conditions for the test functions.

2. Poisson's Half-Summation

What about Fourier sums (which I will call "half-sums") like

$$S_{\pm}(x):=\sum_{n=\pm1}^{\pm\infty}e^{-2inx} \tag{2.1}$$

arising in various applications ($x\to\pi x/\lambda$ gives the general form)? Can they also be expressed similarly?

The answer is YES and takes the form

$$S_{+}(x)=\sum_{n=1}^{\infty}e^{-2inx}=\frac{\pi}{2}\sum_{m=-\infty}^{+\infty}\delta\left(x-m\pi\right)-\frac{1}{2}-\frac{i}{2}\mathrm{P.}V.\cot x, \tag{2.2}$$

i.e. a constant term and an imaginary principal value (P.V.) has to be added for the half-sum.

Since

$$S_{-}(x):=\sum_{n=-1}^{-\infty}e^{-2inx}\equiv S_{+}^{*}(x)=\frac{\pi}{2}\sum_{m=-\infty}^{+\infty}\delta\left(x-m\pi\right)-\frac{1}{2}+\frac{i}{2}\mathrm{P.}V.\cot x \tag{2.3}$$

The full Poisson summation formula is recovered immediately:

$$S_{-}(x)+\underbrace{S_0}_{=1}+S_{+}(x)=\sum_{n=-\infty}^{+\infty}e^{-2inx}=\pi\sum_{m=-\infty}^{+\infty}\delta\left(x-m\pi\right)$$

Which is Eq. (1.2?) with $\lambda=\pi$

Note that variants of Eq. (2.2) where the summation starts at some integer $N\geq 1$ can be derived easily since

$$\sum_{n=N}^{\infty}e^{-2inx}=\sum_{n=1}^{\infty}e^{-2inx}-\sum_{n=1}^{N-1}e^{-2inx}=S_{+}(x)-\sum_{n=1}^{N-1}e^{-2inx} \tag{2.4}$$

(empty sums are to be set to zero). Compared to the original "half-summation" formula only finite sums are to be evaluated additionally. Alternatively, one may shift the summation index to obtain

$$\sum_{n=N}^{\infty}e^{-2inx}\overset{(n'=n-N+1)}{=}e^{-2i(N-1)x}\sum_{n'=1}^{\infty}e^{-2in'x}=e^{-2i(N-1)x}S_{+}(x). \tag{2.5}$$

3. Proof of Eq. (2.2)

To prove Poisson's half-summation formula I will evaluate the Fourier series as a geometric series[2] regularized with a factor $\exp(-2n\eta)$ $\eta\to 0$ to make it convergent:

$$S_{+}(x)=\sum_{n=1}^{\infty}\exp\left[-2in(x-i\eta)\right]=\frac{\exp(-2i(x-i\eta))}{1-\exp(-2i(x-i\eta))}=\left[\exp(2i(x-i\eta))-1\right]^{-1}$$

$$=\frac{(1+\epsilon)\cos 2x-1-i(1+\epsilon)\sin 2x}{\epsilon^2+4(1+\epsilon)\sin^2 x}$$

with $\epsilon=e^{2\eta}-1$ and using $\cos 2x=1-2\sin^2 x$

$$=\frac{\epsilon}{\epsilon^2+4(1+\epsilon)\sin^2 x}-\frac{2(1+\epsilon)\sin^2 x}{\epsilon^2+4(1+\epsilon)\sin^2 x}$$
$$-i\frac{(1+\epsilon)\sin 2x}{\epsilon^2+4(1+\epsilon)\sin^2 x}:=S_1(x)+S_2(x)+S_3(x). \tag{3.1}$$

Let us now discuss the different in the limit · → 0: Taking into account the representation of the δ-function (see, e. g. Ref. [4], eq. (A15c))

$$\delta(y-y_0)=\frac{1}{\pi}\lim_{\epsilon\to 0}\frac{\epsilon}{\epsilon^2+(y-y_0)^2} \tag{3.2}$$

The first term tends to the Dirac comb

$$S_1\overset{\epsilon\to 0}{\to}\pi\delta(2\sin x)=\frac{\pi}{2}\sum_{m=-\infty}^{\infty}\delta\left(x-m\pi\right). \tag{3.3}$$

as the extra factor $1+4\epsilon$ doesn't matter in the limit $\epsilon\to 0$. The second term vanishes exactly at $x = 0$ but approaches the constant

$$S_2(x)\overset{\epsilon\to 0}{\to}-\frac{1}{2} \tag{3.4}$$

for decreasing ϵ nearly everywhere (see Figure 1). Thus for smooth test functions, Eq. (3.4) certainly holds.

Finally, the last one tends to Cauchy's *principal value* distribution

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[1]In Oliver Heaviside's spirit: " *Shall I refuse my dinner because I do not fully understand the process of digestion?* " [6] and in a broader sense of *Experimental Mathematics* [7].

[2]Supporting the joke/observation that theoretical physicist only can sum exponential and geometric series...

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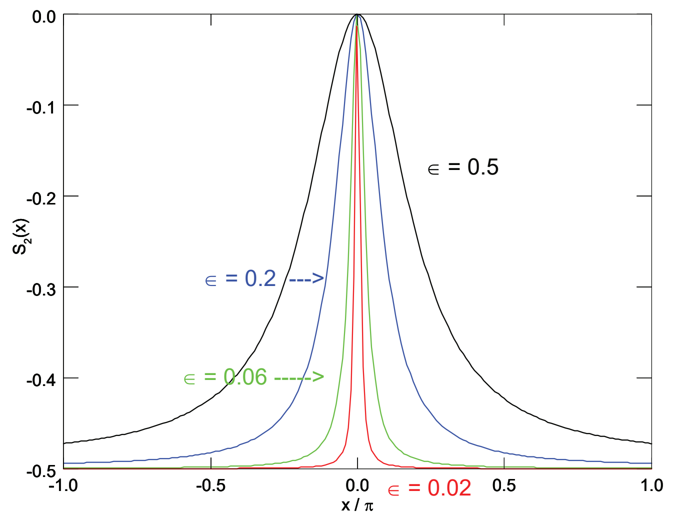

Figure 1: The function $S_2(x) = -2(1+\epsilon)\sin^2 x / [\epsilon^2 + 4(1+\epsilon)\sin^2 x]$ from Eq. (3.1) for different values of the regularization parameter ϵ. It is seen that it can be replaced by the constant -1/2 nearly everywhere.

$$S_3 \overset{\epsilon\to 0}{\to} -i\mathrm{P.V.}\frac{\sin(2x)}{4\sin^2 x} = -i\mathrm{P.V.}\frac{\cos x}{2\sin x} = -\frac{i}{2}\mathrm{P.V.}\cot x. \tag{3.5}$$

defined by

$$\mathrm{P.V.}\int_{-\infty}^{+\infty} dx\, f(x)\cot x := \lim_{\epsilon\to}\left\{\int_{-\infty}^{-\epsilon} dx\, f(x)\cot x + \int_{\epsilon}^{+\infty} dx\, f(x)\cot x\right\}. \tag{3.6}$$

Although this is not immediately obvious, a closer inspection of the integral near x = 0 taken with a smooth test function

$$\Delta S_3 = -\frac{i}{4}\int_{-\epsilon}^{+\epsilon} dx\, f(x)\frac{(1+\epsilon)\sin 2x}{\epsilon^2 + 4(1+\epsilon)\sin^2 x} =$$

$$-\frac{i}{4}\int_{-\epsilon}^{+\epsilon} dx\, f(x)\frac{d}{dx}\ln\left[\epsilon^2 + 4(1+\epsilon)\sin^2 x\right]$$

$$= -\frac{i}{4}\left[f(\epsilon) - f(-\epsilon)\right]\ln\left[\epsilon^2 + 4(1+\epsilon)\sin^2 x)\right] +$$

$$\frac{i}{4}\int_{-\epsilon}^{+\epsilon} dx\ \ln\left[\epsilon^2 + 4(1+\epsilon)\sin^2 x)\right] f'(x) \tag{3.7}$$

Reveals that the ϵ-regularization used in Eq. (1) takes away the diverging contribution in a similar way as in Cauchy's definition, viz.

$$\Delta S_3 \overset{\epsilon\to 0}{\to} \to const.\epsilon \ln\epsilon \to 0 \tag{3.8}$$

If the test function is smooth and regular at the origin. Outside the ϵ-interval around the singularity, the regularization parameter ϵ can then be set to zero without impunity. Additional evidence for the correctness comes from Example 3 which shows explicitly that (at least for some specific test functions) this procedure indeed regularizes the otherwise divergent integral and gives the correct result.

Altogether this yields Poisson's half-summation formula (2.2).

An alternative proof starts from Poisson's full summation formula (1.2) $\lambda = \pi$. Taking the real part[3] gives

$$1 + 2\sum_{n=1}^{\infty}\cos\left(2nx\right) = \pi\sum_{m=-\infty}^{\infty}\delta\left(x - m\pi\right) \tag{3.9}$$

which leads to Eq. (1.6). Now it is well known that the *Hilbert transform* of a cosine function is the negative of a sine function (Ref. [8], eq. 3.722.8 or eq. 15.2.47 in Ref. [9])

$$\mathbb{H}[\cos(\alpha x)](y) := \frac{1}{\pi}\mathrm{P.V.}\int_{-\infty}^{+\infty} dx\,\frac{\cos(\alpha x)}{x-y} = -\sin\left(|\alpha|\, y\right) \tag{3.10}$$

and (up to a sign) vice versa. This can be easily obtained by standard contour deformation in the integral

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} dx\frac{\exp(i\alpha x)}{x-y-i\epsilon} = \frac{1}{\pi}2\pi i\Theta(\alpha)\exp(i\alpha y). \tag{3.11}$$

and the Sokhotski-Plemelj formula (Ref. [4], eq. (A15.e))

$$\lim_{\epsilon\to 0^+}\frac{1}{x-y-i\epsilon} = \mathrm{P.V.}\frac{1}{x-y} + i\pi\delta\left(x-y\right). \tag{3.12}$$

Therefore

$$\mathbb{H}\left[\cos(\alpha x)\right](y) + i\mathbb{H}\left[\sin(\alpha x)\right](y) = -\sin(|\alpha|\, y) + i sgn(\alpha)\cos(\alpha y) \tag{3.13}$$

where $sgn(\alpha) = 2\Theta(\alpha) - 1$ is the sign function. Equating the real parts one obtains Eq. (3.10). This allows transforming the sum over positive integers

$$\sum_{n=1}^{\infty}\sin\left(2ny\right) = -\mathbb{H}\left[\sum_{n=1}^{\infty}\cos\left(2nx\right)\right](y) = \frac{1}{2}\mathbb{H}[-1 + \pi\sum_{m=1}^{\infty}\delta\left(x - m\pi\right)](y) \tag{3.14}$$

and to use the real part (3.9) of Poisson's summation formula. As is well known the Hilbert transform of a constant vanishes:

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[3]Taking the imaginary part gives the trivial relation 0 =0 since the l.h.s. is real and the sum over positive and negative values of the argument of a sine-function vanishes on the r.h.s. If the sum only extends over positive integers this does not hold anymore. $\lambda = \pi$

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$$\mathbb{H}[const](y) = \lim_{\substack{\epsilon\to 0^+\\ R\to\infty}} \left\{ \int_{-R}^{y-\epsilon} dx \frac{const}{x-y} + \int_{y+\epsilon}^{R} dx \frac{const}{x-y} \right\} =$$

$$\lim_{\substack{\epsilon\to 0^+\\ R\to\infty}} const\{\ln|x-y|_{-R}^{y-\epsilon} + \ln|x-y|_{y+\epsilon}^{R}\}$$

$$= const \lim_{\substack{\epsilon\to 0^+\\ R\to\infty}} \{\ln\epsilon - \ln|R+y| + \ln|R-y| - \ln\epsilon\} =$$

$$const \lim_{R\to\infty} \ln\left|\frac{R-y}{R+y}\right| = 0. \tag{3.15}$$

Thus

$$\sum_{n=1}^{\infty} \sin(2ny) = -\frac{\pi}{2}\sum_{m=-\infty}^{\infty} \mathbb{H}\left[\delta(x-m\pi)\right](y) = \frac{1}{2}\mathrm{P.}V.\sum_{m=-\infty}^{\infty}\frac{1}{y-m\pi}$$

$$= \frac{1}{2}\mathrm{P.}V.\left\{\frac{1}{y} + \frac{2y}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{y^2/\pi^2 - m^2}\right\} = \frac{1}{2}\mathrm{P.}V.\cot y \tag{3.16}$$

where in the last line eq. (1.421.3) in Ref. [8] has been used. Combining Eqs. (3.9) and (3.16) one indeed has

$$S_+(x) = \sum_{n=1}^{\infty} e^{-2inx} = \sum_{n=1}^{\infty}[\cos(2nx) - i\sin(2nx)] = \frac{\pi}{2}\sum_{m=-\infty}^{\infty}\delta(x-m\pi) - \frac{1}{2} - \frac{i}{2}\mathrm{P.}V.\cot x. \tag{3.17}$$

Note that Eqs. (2) and (12) allow writing this in the compact form

$$S_+(x) = -\frac{1}{2}\left[1 + \lim_{\epsilon\to 0^+}\frac{i}{\tan x - i\epsilon}\right] \tag{3.18}$$

4. Tests and examples

1) An even discontinuous integral

Consider the test function $F(x) = \Theta(y^2 - x^2)$. Then

$$I_1(y) := \int_{-y}^{+y} dx S_+(x) = \sum_{n=1}^{\infty}\int_{-y}^{+y} dx \exp(-2ixn). \tag{4.1}$$

Its exact value is obtained from Ref. [8], eq. 1.441.1 after performing the x-integration

$$I_1(y) = \sum_{n=1}^{\infty}\frac{\sin(2ny)}{n} = \frac{1}{2}(\pi - 2y) \quad for \quad 0 < 2y < 2\pi. \tag{4.2}$$

With Poisson's half-summation formula (2.2) we have

$$I_1(y) = \frac{\pi}{2}\underbrace{\sum_{m=-\infty}^{+\infty}\int_{-y}^{+y}dx\delta(x-m\pi)}_{=1, if\ 0<y<\pi\ only\ m=0\ contributes} - \frac{1}{2}2y - \frac{i}{2}\mathrm{P.}V.\underbrace{\int_{-y}^{+y}dx\cot x}_{=0, integrand\ is\ odd} = \frac{\pi}{2} - y \quad correct! \tag{4.3}$$

Note that the integral is discontinuous (sawtooth-like) as a function y. For example, if $y = \pi - 0^+$ (meaning that y is infinitesimally smaller than π) Eq. (4.2) gives $I_1 \to -\pi/2$. However, for $y = \pi$ (where Eq. (4.2) cannot be applied) one sees that in Poisson's half-summation now also the terms with $m = \pm 1$ contributing but only with a factor 1/2. This is because[4]

$$\Theta(\mp\pi - x)\delta(x \pm \pi) = \Theta(0)\delta(x \pm \pi) = \frac{1}{2}\delta(x \pm \pi). \tag{4.4}$$

Thus eq. (2.2) gives

$$I_1(\pi) = \frac{\pi}{2}\left[\frac{1}{2} + 1 + \frac{1}{2}\right] - \frac{1}{2}2\pi = 0$$

$$0.3cm(m=-1) \quad 0.7cm(m=0) \quad 0.7cm(m=+1)$$

which is correct since $\sum_{n=1}^{\infty}\frac{\sin(2n\pi)}{n} = 0$. Finally, for $y = \pi + 0^+$ the terms with $m = \pm 1$ now contribute fully and therefore $I_1 \to +\pi/2$ which is the correct value since $I_1(y)$ is a periodic function of y with period π and therefore

$$I_1(\pi + 0^+) = I_1(0^+) \to \pi/2.$$

2) An odd discontinuous integral

Consider now the test function $F(x) = x\Theta(y^2 - x^2)$. Then

$$I_2(y) := \int_{-y}^{+y} dx\, xS_+(x) = \sum_{n=1}^{\infty}\int_{-y}^{+y} dx x e^{-2inx}. \tag{4.5}$$

After integration the exact value is obtained from eqs. 27.8.1 and 27.8.6 in Ref. [10]

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[4] Of course, this is only a rough calculational rule, as neither exists nor does the product of two distributions. However, for finite regularization any representation of Dirac's distribution like (3.2) gives ½ when integrated up to yo but 1 when integrated over the full interval. I assume that this property also holds in the limit . For a more detailed discussion how products of distributions may be handled see Ref. [11].

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$$I_2(y) = -\frac{i}{2}\underbrace{\sum_{n=1}^{\infty}\frac{\sin(2ny)}{n^2}}_{=f_C(2y)} + iy\underbrace{\sum_{n=1}^{\infty}\frac{\cos(2ny)}{n}}_{=-\ln(2\sin y)} \quad for\, 0 < y < \pi \tag{4.6}$$

where

$$f_C(\theta) = -\int_0^{\theta} dt \ln\left(2\sin\frac{t}{2}\right) = \sum_{n=1}^{\infty}\frac{\sin n\theta}{n^2} \tag{4.7}$$

is Clausen's integral, a special dilogarithm (see ch. 25.12 in Ref. [5]).

Posson's half-summation gives

$$I_2(y) = \frac{\pi}{2}\underbrace{\sum_{m=-\infty}^{+\infty}\int_{-y}^{+y}dx\, x\delta(x-m\pi)}_{=0,\ if\ 0<y<\pi\ only\ m=0\ contributes} - \frac{1}{4}\underbrace{x^2\,\Big|_{-y}^{+y}}_{=0} - \frac{i}{2}\int_{-y}^{+y}dx\, x\cot x \tag{4.8}$$

Integration by parts in the last integral yields the correct result

$$I_2(y) = -i\int_0^y dx\, x\cot x = -ix\ \ln(2\sin x)\Big|_0^y + i\int_0^y dx\ \ln(2\sin x)$$

$$= -iy\ln(2\sin y) - \frac{i}{2}f_C(2y). \tag{4.9}$$

3) A check of the principal value prescription

In Example 1 the (assumed) principal value integral did not contribute since the test function was symmetric over the symmetric interval whereas in Example 2 the singularity of the integrand at $x=0$ was removed. In both cases, the Cauchy principal value prescription of the diverging integral was *not* tested. The simplest way to do so is to integrate over the singularity in an asymmetric interval, e.g.

$$I_3\left(y_1, y_2\right) := \int_{-y_1}^{+y_2}dx\ S_+(x) = \frac{1}{2i}\sum_{n=1}^{\infty}\left(e^{2iny_1} - e^{-2iny_2}\right)$$

$$with \quad 0 < y_1 < y_2 < \pi. \tag{4.10}$$

The exact value is obtained from the summable series (see, e.g. Ref. [10], eq. 27. 8. 6)

$$\sum_{n=1}^{\infty}\frac{\exp(\pm in\theta)}{n} = -\ln\left[2\sin\left(\frac{\theta}{2}\right)\right] \pm \frac{i}{2}\left(\pi - \theta\right) \quad 0 < \theta < 2\pi \tag{4.11}$$

as

$$I_3\left(y_1, y_2\right) = \frac{\pi}{2} - \frac{1}{2}\left(y_1 + y_2\right) + \frac{1}{2i}\ln\left(\frac{\sin y_2}{\sin y_1}\right). \tag{4.12}$$

Poisson's half-summation formula gives

$$I_3\left(y_1, y_2\right) = \frac{\pi}{2} - \frac{1}{2}\left(y_1 + y_2\right) - \frac{i}{2}\mathrm{P.V.}\int_{-y_1}^{+y_2}dx\cot x \tag{4.13}$$

Where the last integral would be divergent $x=0$ since $\cot x = 1/x - x/3 + O(x^3)$ at small values of x. However, with Cauchy's principal value prescription, we have to evaluate Eq. (6), i.e.

$$\mathrm{P.V.}\int_{-y_1}^{+y_2}dx\cot x = \lim_{\epsilon\to 0}\left[\int_{-y_1}^{-\epsilon}dx\cot x + \int_{+\epsilon}^{y_2}dx\cot x\right]. \tag{4.14}$$

The indefinite integral is standard (see, e.g. Ref. [12], eq. 453.11)

$$\int dx\ \cot x = \int dx\frac{\cos x}{\sin x} = \int d(\sin x)\frac{1}{\sin x} = \ln|\sin x| \tag{4.15}$$

so that

$$\mathrm{P.V.}\int_{-y_1}^{+y_2}dx\cot x = \lim_{\epsilon\to 0}\left[\ln|\sin x|\,\Big|_{-y_1}^{-\epsilon} + \ln|\sin x|\,\Big|_{\epsilon}^{+y_2}\right]$$

$$= \lim_{\epsilon\to 0}\left[\ln\left(\sin\epsilon\right) - \ln\left(\sin y_1\right) + \ln\left(\sin y_2\right) - \ln\left(\sin\epsilon\right)\right]$$

$$= \ln\left(\frac{\sin y_2}{\sin y_1}\right), \tag{4.16}$$

i.e. the divergent parts have been canceled and the correct result is obtained.

Summary

I have "derived" (for a mathematician only "made plausible") a Poisson-like summation formula that allows evaluation (or rather transformation) of "half-sums", i.e. sums from $n=1$ (or some finite integer) to ∞. Simple examples were used to validate its outcome. Still, some (mostly mathematical) questions remain, e.g. about the allowed class of test functions and the correct treatment of singularities. These are outside the scope of this short note.

Another question is whether the "half-summation" form (2) has already appeared explicitly in the literature. While this is unknown to me – the older "Handbook" [11] does not contain the Poisson summation formula at all, whereas the newer one [5] only lists the standard form in eq. 1.8.14 – it may be quite possible given the long history of the subject. In any case, the extension discussed here should be a useful addition to the toolbox of the theoretical physicist.

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