



Review Article

On the Bogolubov's chain of kinetic equations, the invariant subspaces and the corresponding Dirac type reduction

Yarema A Prykarpatsky¹, Radoslaw Kycia² and Anatolij K Prykarpatski^{2*}

¹Department of Applied Mathematics, University of Cracow, Krakow, Poland

²Department of Informatics and Telecommunication, Cracow University of Technology, Poland

Received: 16 August, 2021

Accepted: 11 October, 2021

Published: 14 October, 2021

*Corresponding authors: Anatolij K Prykarpatski, Department of Informatics and Telecommunication, Cracow University of Technology, Poland, Tel: +48-535-531-185; Fax: +48 - 12 628-31-58; E-mail: pryk.anat@cybergal.com

ORCID: <https://orcid.org/0000-0001-5124-5890>

Keywords: Multi-particle distribution functions; Boltzmann kinetic equations; Bogolubov kinetic equations chain; Wigner transformation; Functional equations; Canonical Poisson bracket; Lie-poisson brackets; Dirac reduction theory; Invariant functional submanifolds

Copyright: © 2021 Prykarpatsky YA, et al. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.

<https://www.peertechzpublications.com>



Abstract

We study a special class of dynamical systems of Boltzmann-Bogolubov and Boltzmann-Vlasov type on infinite dimensional functional manifolds modeling kinetic processes in manyparticle media. Based on geometric properties of the manyparticle phase space we succeeded in dual analysing of the infinite Bogolubov hierarchy of manyparticle distribution functions and their Hamiltonian structure. Moreover, we proposed a new approach to invariant reducing the Bogolubov hierarchy on a suitably chosen correlation function constraint and deducing the related modified Boltzmann-Bogolubov kinetic equations on a finite set of multiparticle distribution functions.

Kinetic equations, their algebraic structure and invariant reductions

Introduction

It is well known that the classical Bogolubov-Boltzmann kinetic equations under the condition of manyparticle correlations [1-12] at weak short range interaction potentials describe long waves in a dense gas medium. The same equation, called the Vlasov one, as it was shown by N. Bogolubov [5], describes also exact microscopic solutions of the infinite Bogolubov chain [4] for the manyparticle distribution functions, which was widely studied making use of both classical approaches in [2,6,11,13-23] and in [24-32], making use of the generating Bogolubov functional method and the related quantum current algebra representations.

A.A. Vlasov proposed his kinetic equation [33] for electron-ion plasma, based on general physical reasonings, that in contrast to the short range interaction forces between neutral gas atoms, interaction forces between charged particles slowly decrease with distance, and therefore the motion of each such particle is determined not only by its pair-wise interaction with either particle, yet also by the interaction with the whole ensemble of charged particles. In this case the Bogolubov equation for distribution functions in a domain $\Lambda \subset \mathbb{R}^3$

$$\frac{\partial f_1(z;t)}{\partial t} + \left\langle \frac{p}{m} \mid \nabla_x f_1(z;t) \right\rangle = \int_{T^*(\Lambda)} dz' \{f_2(z, z'; t), V(x-x')\}^{(2)}, \quad (1.1)$$



where $z := (x, p) \in T^*(\Lambda), t \in \mathbb{R}_+$ is the temporal evolution parameter, $\{\cdot, \cdot\}^{(m)}$ denotes the canonical Poisson bracket [6,33,34] on the product $T^*(\Lambda)^m, m \in \mathbb{N}$, and $V(x-x'), x, x' \in \Lambda$, is an interparticle interaction potential, - reduces to the Vlasov equation if to put in (1.1)

$$f_2(z, z'; t) = f_1(z; t)f_1(z'; t), \tag{1.2}$$

that is to assume that the two-particle correlation function [2,3,11,23] vanishes:

$$g_2(z, z'; t) = f_2(z, z'; t) - f_1(z; t)f_1(z'; t) = 0 \tag{1.3}$$

for all $z, z' \in T^*(\Lambda)$ and $t \in \mathbb{R}_+$. Then one easily obtains from (1.1) that

$$\frac{\partial f_1(z; t)}{\partial t} + \left\langle \frac{p}{m} \mid \nabla_x f_1(z; t) \right\rangle = \left\langle \frac{\partial f_1(z; t)}{\partial p} \mid \nabla_x \int_{T^*(\Lambda)} dz' V(x-x') f_1(z'; t) \right\rangle \tag{1.4}$$

for all $z \in T^*(\Lambda)$ and $t \in \mathbb{R}_+$. Remark here that the equation (1.4) is reversible under the time reflection $\mathbb{R}_- \ni -t \rightleftharpoons t \in \mathbb{R}_+$, thus it is obvious that it can not describe thermodynamically stable limiting states of the particle system in contrast to the classical Bogolubov-Boltzmann kinetic equations [1,2,4,6,11,24,27], being *a priori* time nonreversible owing to the choice of boundary conditions in the correlation weakening form. This means that in spite of the Hamiltonicity of the Bogolubov chain for the distribution functions, the Bogolubov-Boltzmann equation *a priori* is not reversible. It is also evident that the condition (1.3) does not break the Hamiltonicity - the equation (1.4) is Hamiltonian with respect to the following Lie-Poisson-Vlasov bracket:

$$\{\{a(f), b(f)\}\} := \int_{T^*(\Lambda)} dz f(z) \{grad a(f)(z), grad b(f)(z)\}^{(1)}, \tag{1.5}$$

where $grad(\cdot) := \delta(\cdot) / \delta f, f \in D(T^*(\Lambda)) := M_{f_1}$, respectively $a, b \in D(M_{f_1})$ are smooth functionals on the functional manifold M_{f_1} , consisting of functions fast decreasing at the boundary $\partial\Lambda$ of the domain $\Lambda \subset \mathbb{R}^3$. The statement above easily ensues from the following proposition.

Proposition 1.1 *Let $M_{\mathcal{F}}$ denote a set of many-particle distribution functions. Then the classical Bogolubov-Poisson bracket [4,18,24,25] on the functional space $D(M_{\mathcal{F}})$ reduces invariantly on the subspace $D(M_{f_1}) \subset D(M_{\mathcal{F}})$ to the Lie-Poisson-Vlasov bracket (1.5).*

Concerning the general case when we are work with an innite Bogolubov chain of kinetic equations on the many-particle distribution functions and forced to break it at some place, numbered by some natural number $N \in \mathbb{N}$; the usual approaches always give rise to the resulting inconsistency [3,5] of the chain and, as a result, to the nonphysical solutions. The most successful approach to obtaining the Boltzmann kinetic equation for the one-particle distribution function was suggested still many years ago by N. Bogolubov [1,2], based on the e/ective application of the so called weak correlation condition. So far, to the regret, this approach, being conjugated with the complex problem of solving functional equations, also gives rise to the inconsistency of the higher order kinetic equations. Nonetheless, being inspired by former studies [6, 16, 11] of these problems, based on the geometrical interpretation of the Bogolubov kinetic equations chain, we devised a new functional analytic approach to constructing its compatible reduction *a priori* free of any unphysical consequences. We also succeeded in constructing a reduced set of kinetic equations, based on a suitably devised Dirac type invariant reduction scheme of the corresponding many-particle Lie-Poisson phase space. The approach to solving this problem and its di/erent consequences will be analyzed in more detail in sections to follow below.

The Lie-Poisson-Vlasov bracket: Lie-algebraic approach

The bracket expression (1.5) allows a slightly different Lie-algebraic interpretation, based on considering the functional space $D(M_{f_1})$ as a Poissonian manifold, related with the canonical symplectic structure on the diffeomorphism group $Diff(\Lambda)$ of the domain $\Lambda \subset \mathbb{R}^3$, first described [35,36] still in 1887 by Sophus Lie. Namely, the following classical theorem holds.

Theorem 1.2 *The Lie-Poisson bracket at point $(\mu; \eta) \in T_\eta^*(Diff(\Lambda))$ on the coadjoint space $T_\eta^*(Diff(\Lambda)), \eta \in Diff(\Lambda)$, is equal to the expression*

$$\{f, g\}(\mu) = (\mu \mid [\delta g(\mu) / \delta \mu, \delta f(\mu) / \delta \mu])_c \tag{1.6}$$

for any smooth right-invariant functionals $f, g \in C^\infty(T_\eta^*(Diff(\Lambda)); \mathbb{R})$.



Proof. By classical definition [33-37] of the Poisson bracket of smooth functions $(\mu|a)_c, (\mu|b)_c \in C^\infty(T_\eta^*(Diff(\Lambda)); \mathbb{R}), a, b \in diff(\Lambda) \simeq T_\eta(Diff(\Lambda))$ on the symplectic space $T_\eta^*(Diff(\Lambda))$, it is easy to calculate that

$$\begin{aligned} \{\mu(a), \mu(b)\} &:= \delta\alpha(X_a, X_b) = \\ &= X_a(\alpha|X_b)_c - X_b(\alpha|X_a)_c - (\alpha|[X_a, X_b])_c, \end{aligned} \tag{1.7}$$

where $X_a := \delta(\mu|a)_c / \delta\mu = a \in diff(\Lambda), X_b := \delta(\mu|b)_c / \delta\mu = b \in diff(\Lambda)$. Since the expressions $X_a(\alpha|X_b)_c = 0$ and $X_b(\alpha|X_a)_c = 0$ owing the right-invariance of the vector fields $X_a, X_b \in T_\eta(Diff(\Lambda))$, the Poisson bracket (1.7) transforms into

$$\begin{aligned} \{(\mu|a)_c, (\mu|b)_c\} &= -(\alpha|[X_a, X_b])_c = \\ &= (\mu|[b, a])_c = (\mu|[\delta(\mu|b)_c / \delta\mu, \delta(\mu|a)_c / \delta\mu])_c \end{aligned} \tag{1.8}$$

for all $(\mu; \eta) \in T_\eta^*(Diff(\Lambda)) \simeq diff^*(\Lambda)$, and any $a, b \in diff(\Lambda)$. The Poisson bracket (1.8) is easily generalized to

$$\{f, g\}(\mu) = (\mu|[\delta g(\mu) / \delta\mu, \delta f(\mu) / \delta\mu])_c \tag{1.9}$$

for any smooth functionals $f, g \in C^\infty(diff^*(\Lambda); \mathbb{R})$, finishing the proof.

Concerning our special problem of describing evolution equations for one-particle distribution functions, we will consider the one particle cotangent space $T^*(\Lambda)$ over a domain $\Lambda \subset \mathbb{R}^3$ and the canonical Poisson bracket $\{\cdot, \cdot\} := \{\cdot, \cdot\}^{(1)}$ on $T^*(\Lambda)$, for which, by definition, for any $f, g \in M_{f_1}$

$$\{f, g\}(z) := \left\langle \frac{\partial f}{\partial p} \middle| \frac{\partial g}{\partial x} \right\rangle - \left\langle \frac{\partial g}{\partial p} \middle| \frac{\partial f}{\partial x} \right\rangle, \tag{1.10}$$

where $z = (x, p) \in T^*(\Lambda)$. Denote now by $\mathcal{G} := (M_{f_1}; \{\cdot, \cdot\})$ the related functional Lie algebra and \mathcal{G}^* its adjoint space with respect to the standard bilinear symmetric form $(\cdot|\cdot) : M_{f_1} \times M_{f_1} \rightarrow \mathbb{R}$ on the product $M_{f_1} \times M_{f_1}$, where

$$(f|g) := \int_{T^*(\Lambda)} f(z)g(z)dz. \tag{1.11}$$

The constructed Lie algebra \mathcal{G} with respect to the bilinear symmetric form (1.11) proves to be metrized, that is $\mathcal{G} \simeq \mathcal{G}^*$ and

$$(\{f, g\}|h) = (f|\{g|h\}) \tag{1.12}$$

for any f, g and $h \in \mathcal{G}$. If $\gamma \in D(\mathcal{G}^*)$ is a smooth functional on \mathcal{G}^* , its gradient $grad\gamma(f) \in \mathcal{G}$ at point $f \in \mathcal{G}^*$ is naturally defined via the limiting expression

$$(g|grad\gamma(f)) := \left. \frac{d}{d\varepsilon} \gamma(f + \varepsilon g) \right|_{\varepsilon=0} \tag{1.13}$$

for arbitrary element $g \in \mathcal{G}^*$. Define now the Poisson structure $\{\{\cdot, \cdot\}\} : \mathcal{G}^* \times \mathcal{G}^* \rightarrow \mathbb{R}$ by means of the standard Lie-Poisson [9,33,34-36,38,39] expression:

$$\{\{\gamma, \mu\}\} := (f|\{\{grad\gamma(f), grad\mu(f)\}\}) \tag{1.14}$$

for arbitrary functionals $\gamma, \mu \in D(\mathcal{G}^*)$. It is evident that the expression (1.14) identically coincides with the Poisson bracket (1.5).

Consider a functional $\gamma \in D(\mathcal{G}^*)$ and the related coadjoint action of the element $grad\gamma(f) \in \mathcal{G}$ at a fixed element $f := f_1 \in \mathcal{G}^*$:



$$\partial f_1 / \partial t := ad_{grad \gamma(f_1)}^* f_1, \tag{1.15}$$

where $t \in \mathbb{R}$ is the corresponding evolution parameter. It is easy observe that

$$\partial f_1 / \partial t = \{ \gamma, f_1 \} \tag{1.16}$$

is a Hamiltonian equation with the functional $\gamma \in D(\mathcal{G}^*)$ taken as its Hamiltonian, being simultaneously equivalent to the following canonical Hamiltonian flow:

$$\partial f_1 / \partial t = \{ f_1, grad \gamma(f_1) \}, \tag{1.17}$$

if to choose as a Hamiltonian the following functional

$$\gamma(f_1) := \int_{T^*(\Lambda)} dz_1 \frac{p_1^2}{2m} f_1(z_1) + \frac{1}{2} \int_{T^*(\Lambda)^2} dz_1 dz_2 V(x_1 - x_2) f_1(z_1) f_1(z_2), \tag{1.18}$$

where $V(x_1 - x_2)$ is a two-particle interaction potential, $x_1, x_2 \in \Lambda$. It is easy to observe here that the Hamiltonian (1.18) is obtained from the corresponding classical Bogolubov Hamiltonian expression

$$\mathcal{H}(\mathcal{F}) := \int_{T^*(\Lambda)} dz_1 \frac{p_1^2}{2m} f_1(z_1) + \frac{1}{2} \int_{T^*(\Lambda)^2} dz_1 dz_2 V(x_1 - x_2) f_2(z_1, z_2), \tag{1.19}$$

where $\mathcal{F} = (f_1, f_2, \dots) \in M_{\mathcal{F}}$ denotes an infinite vector from the space $M_{\mathcal{F}} := \prod_{j \in \mathbb{N}} M_{f_j}$ of multiparticle distribution functions, and if to impose on it the constraint (1.2). Thus we have stated the following proposition.

Proposition 1.3 *The Boltzmann-Vlasov kinetic equation (1.4) is a Hamiltonian system on the functional manifold $\mathcal{G}^* \simeq \mathcal{G} = (M_{\mathcal{F}}; \{ \cdot, \cdot \})$ with respect to the canonical Lie-Poisson structure (1.14) with Hamiltonian (1.18). As a consequence, the flow (1.4) is time reversible.*

Boltzmann-Vlasov kinetic equations and microscopic exact solutions

Proposition 1.1, stated above, claims that the Boltzmann-Vlasov equation (1.4) is a suitable reduction of the whole Bogolubov chain upon the invariant functional subspace $M_{f_1} \subset M_{\mathcal{F}}$. Moreover, this invariance in no way should be compatible *a priori* [5,19,21,24,25,27] with the other kinetic equations from the Bogolubov chain, and can be even contradictory. Nonetheless, as it was stated [5] by N. Bogolubov, namely owing to this invariance of the subspace $M_{f_1} \subset M_{\mathcal{F}}$ the Boltzmann-Vlasov equation (1.4) in the case of the Boltzmann-Enskog hard sphere approximation of the inter-particle potential possesses exact microscopical solutions which are compatible with the whole hierarchy of the Bogolubob kinetic equations. The latter is, obviously, equivalent to its Hamiltonicity on the manifold M_{f_1} with respect to the Lie-Poisson bracket (1.14). The Boltzmann-Enskog kinetic equation [3,5,11,12,23] equals

$$\begin{aligned} & \frac{\partial f_1(z;t)}{\partial t} + \left\langle \frac{p}{m} \mid \nabla_x f_1(z;t) \right\rangle = \\ & = a^2 \int_{\mathbb{S}^2} dn \int_{\mathbb{B}^3} dp' \langle p' \mid n \rangle \left\langle \frac{\tilde{p}'}{m} \mid n \right\rangle \left[f_2(x, \tilde{p}; x + an, \tilde{p}'; t) - f_2(x, p; x - an, p'; t) \right] \end{aligned} \tag{1.20}$$

where $\tilde{p} := p + n \langle p' - p \mid n \rangle$, $\tilde{p}' := p - n \langle p' - p \mid n \rangle$, $a > 0$ – a particle diameter, $n \in \mathbb{S}^2$ – a unit vector, $\langle n \mid n \rangle = 1$, and, by definition, $f_2(z, z'; t) = 0$ for all $z, z' \in T^*(\Lambda)$, $t \in \mathbb{R}$, satisfying the condition $\|z - z'\| < a$. The equation (1.20) easily reduces to the Vlasov-Enskog equation



$$\frac{\partial f_1(z;t)}{\partial t} + \left\langle \frac{p}{m} \mid \nabla_x f_1(z;t) \right\rangle = J_{V-E}(f),$$

$$J_{V-E}(f) = a^2 \int_{\mathbb{S}^2} dn \int_{\mathbb{B}^3} dp' \langle p' \mid n \rangle \left\langle \frac{\tilde{p}'}{m} \mid n \right\rangle \times$$

$$\times \left[f_1(x, \tilde{p}; t) f_1(x + an, \tilde{p}'; t) - f_1(x, p; t) f_1(x - an, p'; t) \right] \tag{1.21}$$

for all $(z;t) \in T^*(\Lambda) \times \mathbb{R}$ owing to its Hamiltonicity on the space $M_{f_1} \subset M_{\mathcal{F}}$. If in addition there exists a nontrivial interparticle potential, the equation above is naturally generalized to the kinetic equation

$$\frac{\partial f_1(z;t)}{\partial t} + \left\langle \frac{p}{m} \mid \nabla_x f_1(z;t) \right\rangle = J_{V-E}(f) +$$

$$+ \int_{T^*(\Lambda)} dz' \{ f_1(z;t) f_1(z';t), V(x-x') \}^{(2)}, \tag{1.22}$$

which remains to be Hamiltonian on M_{f_1} and possesses, in particular, the following exact singular solution:

$$f_1(z;t) = \sum_{j=1,N} \delta(z - z_j(t)), \tag{1.23}$$

where $z_j(t) \in T^*(\Lambda), j = \overline{1,N}$, - phase space coordinates in $T^*(\Lambda)^N$ of $N \in \mathbb{N}$ interacting particles in the domain $\Lambda \subset \mathbb{R}^3$. Specified above the Hamiltonicity problem and the existence of exact solutions to the Boltzmann-Vlasov kinetic equation (1.22) is deeply related to that of describing correlation functions [2,11,23], suitably breaking the infinite Bogolubov chain [2,4,11,24,30,31] of manyparticle distribution functions. Namely, if to introduce manyparticle correlation functions [2,11,23] for related Bogolubov distribution functions as

$$g_1(z_1) = 0, g_2(z_1, z_2) = f_2(z_1, z_2) - f_1(z_1) f_1(z_2), \tag{1.24}$$

$$g_3(z_1, z_2, z_3) = f_3(z_1, z_2, z_3) - f_1(z_1) f_1(z_2) f_1(z_3) - f_1(z_1) g_2(z_2, z_3) -$$

$$- f_1(z_2) g_2(z_3, z_1) - f_1(z_3) g_2(z_1, z_2), \dots,$$

where $z_j \in T^*(\Lambda), j \in N$, then the Vlasov equation (1.22) is obtained from the Bogolubov hierarchy at $n=1$ and $g_2(z_1, z_2) = 0$ for all $z_1, z_2 \in T^*(\Lambda)$.

As it was mentioned above, the constraint imposed on the infinite Bogolubov hierarchy is compatible with its Hamiltonicity. Yet in many practical cases this closedness procedure by means of imposing the conditions like

$$g_{m+1}(z_1, z_2, \dots, z_{m+1}) = 0 \tag{1.25}$$

for all $z_s \in T^*(\Lambda), s = \overline{1, m+1}$ at some fixed $m \geq 2$ gives rise to some serious dynamical problems related with its mathematical correctness. Namely, if to close this way the infinite Bogolubov chain of kinetic equations on manyparticle distribution functions, one easily checks that the imposed constraint (1.25) does not persists in time subject to the evolution of the distribution functions $f_j(z_1, z_2, \dots, z_j), z_j \in T^*(\Lambda), j = \overline{1, m}$. This means that these naively reduced kinetic equations are written down somehow incorrectly, as the reduced functional submanifold $M_{\mathcal{F}}^{(m)} := \{ \mathcal{F} \in M_{\mathcal{F}} : g_{m+1} = 0 \}$ should remain invariant in time. To dissolve this problem we are forced to consider the whole Bogolubov hierarchy of kinetic equations on multiparticle distribution functions as a Hamiltonian



system on the functional manifold $M_{\mathcal{F}}$ and correctly reduce it on the constructed above functional submanifold $M_{\mathcal{F}}^{(m)} \subset M_{\mathcal{F}}$ via the classical Dirac type [1, 3, 6, 19, 36] procedure. The kinetic equations obtained this way by means of the reduced Lie-Poisson-Bogolubov structure will evidently differ from those naively obtained by means of the direct substitution of the imposed constraint (1.25) into the Bogolubov chain of kinetic equations, and in due course will conserve the functional submanifold $M_{\mathcal{F}}^{(m)} \subset M_{\mathcal{F}}$ invariant.

The invariant reduction of the Bogolubov distribution functions chain

Consider the constructed before Hamiltonian functional $\mathcal{H}(\mathcal{F}) \in D(M_{\mathcal{F}})$ (1.19)

$$\mathcal{H}(\mathcal{F}) = \int_{T^*(\Lambda)} dz_1 \frac{p_1^2}{2m} f_1(z_1) + \frac{1}{2} \int_{T^*(\Lambda)^2} dz_1 dz_2 V(x_1 - x_2) f_2(z_1, z_2) \tag{1.26}$$

and calculate the evolution of the distribution functions vector $\mathcal{F} \in M_{\mathcal{F}}$ under the simplest constraint (1.25) at $m = 1$, that is

$$g_2(z_1, z_2) = f_2(z_1, z_2) - f_1(z_1)f_1(z_2) = 0 \tag{1.27}$$

for all $z_1, z_2 \in T^*(\Lambda)$. To perform this reduction on $M_{\mathcal{F}}^{(1)} \subset M_{\mathcal{F}}$ we need [39-43] to constraint the λ -extended Hamiltonian expression

$$\mathcal{H}_{\lambda}(\mathcal{F}) := \mathcal{H}(\mathcal{F}) + \frac{1}{2} \int_{T^*(\Lambda)^2} dz_1 dz_2 \lambda(z_1, z_2) [f_2(z_1, z_2) - f_1(z_1)f_1(z_2)] \tag{1.28}$$

for some smooth function $\lambda \in D(T^*(\Lambda)^2)$ and next to determine it from the submanifold $M_{\mathcal{F}}^{(1)}$ invariance condition

$$\begin{aligned} \frac{\partial g_2(z_1, z_2)}{\partial t} &= \{ \{ \mathcal{H}_{\lambda}(\mathcal{F}), g_2(z_1, z_2) \} \} = \\ &= \frac{\partial f_2(z_1, z_2)}{\partial t} - \frac{\partial f_1(z_1)}{\partial t} f_1(z_2) - f_1(z_1) \frac{\partial f_1(z_2)}{\partial t} = 0 \end{aligned} \tag{1.29}$$

for all $z_1, z_2 \in T^*(\Lambda)$ and $t \in \mathbb{R}$. To calculate effectively the condition (1.29) let us first calculate the evolutions for distribution functions f_1 and $f_2 \in M_{\mathcal{F}}$:

$$\begin{aligned} \frac{\partial f_1(z_1)}{\partial t} &= \{ \{ \mathcal{H}_{\lambda}(\mathcal{F}), f_1(z_1) \} \} = \left\{ f_1(z_1), \frac{\delta \mathcal{H}_{\lambda}(\mathcal{F})}{\delta f_1(z_1)} \right\}^{(1)} + \\ &+ \int_{T^*(\Lambda)} dz_2 \left\{ f_2(z_1, z_2), \frac{\delta \mathcal{H}_{\lambda}(\mathcal{F})}{\delta f_2(z_1, z_2)} \right\}^{(1)}, \end{aligned} \tag{1.30}$$

and

$$\begin{aligned} \frac{\partial f_2(z_1, z_2)}{\partial t} &= \{ \{ \mathcal{H}_{\lambda}(\mathcal{F}), f_2(z_1, z_2) \} \} = \left\{ f_2(z_1, z_2), \frac{\delta \mathcal{H}_{\lambda}(\mathcal{F})}{\delta f_1(z_1)} + \frac{\delta \mathcal{H}_{\lambda}(\mathcal{F})}{\delta f_1(z_2)} \right\}^{(2)} + \\ &+ \left\{ f_2(z_1, z_2), \frac{\delta \mathcal{H}_{\lambda}(\mathcal{F})}{\delta f_2(z_1, z_2)} \right\}^{(2)} + \int_{T^*(\Lambda)} dz_3 \left\{ f_3(z_1, z_2, z_3), \frac{\delta \mathcal{H}_{\lambda}(\mathcal{F})}{\delta f_2(z_1, z_3)} + \frac{\delta \mathcal{H}_{\lambda}(\mathcal{F})}{\delta f_2(z_2, z_3)} \right\}^{(2)}, \end{aligned} \tag{1.31}$$

which can be rewritten equivalently as follows:



$$\begin{aligned}
 \frac{\partial f_1(z_1)}{\partial t} = & -\left\langle \frac{\partial f_1(z_1)}{\partial p_1} \middle| \int_{T^*(\Lambda)} dz_2 \frac{\partial \lambda(z_1, z_2)}{\partial x_1} f_1(z_2) \right\rangle - \\
 & -\left\langle \frac{p_1}{m} - \int_{T^*(\Lambda)} dz_2 \frac{\partial \lambda(z_1, z_2)}{\partial p_1} f_1(z_2) \middle| \frac{\partial f_1(z_1)}{\partial x_1} \right\rangle + \\
 & + \frac{1}{2} \int_{T^*(\Lambda)} dz_2 \left\langle \frac{\partial}{\partial x_1} [V(x_1 - x_2) + \lambda(z_1, z_2)] \middle| \frac{\partial f_2(z_1, z_2)}{\partial p_1} \right\rangle - \\
 & - \frac{1}{2} \int_{T^*(\Lambda)} dz_2 \left\langle \frac{\partial \lambda(z_1, z_2)}{\partial p_1} \middle| \frac{\partial f_2(z_1, z_2)}{\partial x_1} \right\rangle
 \end{aligned}
 \tag{1.32}$$

and

$$\begin{aligned}
 \frac{\partial f_2(z_1, z_2)}{\partial t} = & -\left\langle \frac{\partial f_2(z_1, z_2)}{\partial p_1} \middle| \int_{T^*(\Lambda)} dz_2 \frac{\partial \lambda(z_1, z_2)}{\partial x_1} f_1(z_2) \right\rangle - \\
 & -\left\langle \frac{\partial f_2(z_1, z_2)}{\partial p_2} \middle| \int_{T^*(\Lambda)} dz_1 \frac{\partial \lambda(z_1, z_2)}{\partial x_2} f_1(z_1) \right\rangle - \\
 & -\left\langle \frac{\partial f_2(z_1, z_2)}{\partial x_1} \middle| \frac{p_1}{m} - \int_{T^*(\Lambda)} dz_2 \frac{\partial \lambda(z_1, z_2)}{\partial p_1} f_1(z_2) \right\rangle - \\
 & -\left\langle \frac{\partial f_2(z_1, z_2)}{\partial x_2} \middle| \frac{p_2}{m} - \int_{T^*(\Lambda)} dz_1 \frac{\partial \lambda(z_1, z_2)}{\partial p_2} f_1(z_1) \right\rangle + \\
 & + \frac{1}{2} \left\langle \frac{\partial f_2(z_1, z_2)}{\partial p_1} \middle| \frac{\partial}{\partial x_1} [V(x_1 - x_2) + \lambda(z_1, z_2)] \right\rangle + \\
 & + \frac{1}{2} \left\langle \frac{\partial f_2(z_1, z_2)}{\partial p_2} \middle| \frac{\partial}{\partial x_2} [V(x_1 - x_2) + \lambda(z_1, z_2)] \right\rangle - \\
 & - \frac{1}{2} \left\langle \frac{\partial f_2(z_1, z_2)}{\partial x_1} \middle| \frac{\partial \lambda(z_1, z_2)}{\partial p_1} \right\rangle - \frac{1}{2} \left\langle \frac{\partial f_2(z_1, z_2)}{\partial x_2} \middle| \frac{\partial \lambda(z_1, z_2)}{\partial p_2} \right\rangle + \\
 & + \frac{1}{2} \left\langle \int_{T^*(\Lambda)} dz_3 \frac{\partial f_3(z_1, z_2, z_3)}{\partial p_1} \middle| \frac{\partial}{\partial x_1} [V(x_1 - x_3) + \lambda(z_1, z_3)] \right\rangle + \\
 & + \frac{1}{2} \left\langle \int_{T^*(\Lambda)} dz_3 \frac{\partial f_3(z_1, z_2, z_3)}{\partial p_2} \middle| \frac{\partial}{\partial x_2} [V(x_2 - x_3) + \lambda(z_2, z_3)] \right\rangle - \\
 & - \frac{1}{2} \left\langle \int_{T^*(\Lambda)} dz_3 \frac{\partial f_3(z_1, z_2, z_3)}{\partial x_1} \middle| \frac{\partial \lambda(z_1, z_2)}{\partial p_1} \right\rangle - \frac{1}{2} \left\langle \int_{T^*(\Lambda)} dz_3 \frac{\partial f_3(z_1, z_2, z_3)}{\partial x_2} \middle| \frac{\partial \lambda(z_1, z_2)}{\partial p_2} \right\rangle
 \end{aligned}
 \tag{1.33}$$

Having now substituted temporal derivatives (1.32) and (1.33) into the equality (1.29) in their explicit form, one obtains the following functional relationship:



$$\begin{aligned} & \frac{1}{2} \langle f_1(z_2) \frac{\partial f_1(z_1)}{\partial p_1} \Big| \frac{\partial}{\partial x_1} (V(x_1 - x_2) + \lambda(z_1, z_2) - \\ & \quad - \int_{T^*(\Lambda)} dz_3 f_1(z_3) [V(x_1 - x_3) + \lambda(z_1, z_3)]) \rangle + \\ & + \frac{1}{2} \langle f_1(z_1) \frac{\partial f_1(z_2)}{\partial p_2} \Big| \frac{\partial}{\partial x_2} (V(x_2 - x_1) + \lambda(z_2, z_1) - \\ & \quad - \int_{T^*(\Lambda)} dz_3 f_1(z_3) [V(x_2 - x_3) + \lambda(z_2, z_3)]) \rangle = 0, \end{aligned} \tag{1.34}$$

which is satisfied iff

$$\lambda(z_1, z_2) = -V(x_1 - x_2) \tag{1.35}$$

for all $z_1, z_2 \in T^*(\Lambda)$. Taking into account the result (1.35), one easily obtains from the equation (1.32) the invariantly reduced on the submanifold $M_{\mathcal{F}}^{(1)} \subset M_{\mathcal{F}}$ kinetic equation on the one-particle distribution function:

$$\frac{\partial f_1(z_1)}{\partial t} + \langle p_1 / m \Big| \frac{\partial f_1(z_1)}{\partial x_1} \rangle = \langle \frac{\partial f_1(z_1)}{\partial p_1} \Big| \frac{\partial}{\partial x_1} \int_{T^*(\Lambda)} dz_2 f_1(z_2) V(x_1 - x_2) \rangle, \tag{1.36}$$

which can be rewritten in the following compact form:

$$\frac{\partial f_1(z_1)}{\partial t} = \left\{ f_1(z_1), \frac{\delta \tilde{\mathcal{H}}(\mathcal{F})}{\delta f_1(z_1)} \right\}^{(1)}, \tag{1.37}$$

where we put, by definition,

$$\tilde{\mathcal{H}}(\mathcal{F}) := \int_{T^*(\Lambda)} dz_1 \frac{p_1^2}{2m} f_1(z_1) + \frac{1}{2} \int_{T^*(\Lambda)^2} dz_1 dz_2 V(x_1 - x_2) f_1(z_1) f_1(z_2). \tag{1.38}$$

The kinetic equation (1.36) naturally coincides exactly with that obtained before from the naively reduced evolution equation

$$\frac{\partial f_1(z_1)}{\partial t} = \{ \{ \mathcal{H}(\mathcal{F}), f_1(z_1) \} \} \Big|_{M_{\mathcal{F}}^{(1)}} \tag{1.39}$$

on the submanifold $M_{\mathcal{F}}^{(1)} \subset M_{\mathcal{F}}$, as it is globally invariant [18,24] with respect to the classical Lie–Poisson–Bogolubov structure on $M_{\mathcal{F}}$.

The obtained result can be formulated as the following proposition.

Proposition 1.4 *The first coorelation function Dirac type reduction on the functional submanifold $M_{\mathcal{F}}^{(1)} \subset M_{\mathcal{F}}$, formed by relationships (1.27), reduces the corresponding Bogolubov chain of many-particle kinetic equations to the well known classical Vlasov kinetic equation.*

Remark 1.5 *It is worth to mention here that the well known classical Bogolubov approximation of the many-particle distribution functions as $f_n(z_1, z_2, \dots, z_n) := \varphi_n(z_1, z_2, \dots, z; f_1)$, $z_j \in T(\Lambda)$, $j = \overline{2, n}$, with mapping $\varphi_n : (\dots) \times M_{f_1} \rightarrow \mathbb{R}$, $n \in \mathbb{N} \setminus \{1\}$, presenting smooth nonlinear functionals, independent of the temporal parameter $t \in \mathbb{R}_+$, define a suitably different functional submanifold $\tilde{M}_{\mathcal{F}}^{(1)} \subset M_{\mathcal{F}}$, upon which the reduced evolution flow*

$$\frac{\partial f_1(z_1)}{\partial t} = \{ \{ \mathcal{H}(\mathcal{F}), f_1(z_1) \} \} \Big|_{M_{\mathcal{F}}^{(1)}} \tag{1.40}$$



gives rise to a new Boltzmann type kinetic equation, being compatible with evolution equations for higher distribution functions, free of evolution inconsistencies and completely different from that derived before by Bogolubov [4].

The same way as above one can explicitly construct the system of invariantly reduced kinetic equations

$$\frac{\partial f_1(z_1)}{\partial t} = \{ \{ \mathcal{H}(\mathcal{F}), f_1(z_1) \} \} \Big|_{M_{\mathcal{F}}^{(2)}}, \frac{\partial f_2(z_1, z_2)}{\partial t} = \{ \{ \mathcal{H}(\mathcal{F}), f_2(z_1, z_2) \} \} \Big|_{M_{\mathcal{F}}^{(2)}} \quad (1.41)$$

on the submanifold $M_{\mathcal{F}}^{(2)} \subset M_{\mathcal{F}}$, which already is not *a priori* globally invariant with respect to the Hamiltonian evolution flows on $M_{\mathcal{F}}$ and whose detail structure and analysis are postponed to another place. This case

Conclusion

We studied a well known classical problem of constructing a compatible nite-particle reduction of the Bogolubov chain of many-particle distribution functions and analyzed a special class of the related dynamical systems of BoltzmannBogolubov and Boltzmann Vlasov type on infinite dimensional functional manifolds, modeling kinetic processes in many-particle media. Based on the geometric approach, effectively devised to studying the corresponding many-particle Lie-Poisson functional phase space, we succeeded in dual analysis of the infinite Bogolubov hierarchy of many-particle distribution functions and their Hamiltonian structure. Moreover, we proposed a new an effective approach to invariant Dirac type reduction of the Bogolubov hierarchy upon a suitably chosen invariant Poisson subspace endowed and deduced the related modified BoltzmannBogolubov kinetic equations on a finite set of multi-particle distribution functions.

Acknowledgements

Authors would like to convey their warm thanks to Prof. Gerald A. Goldin for many discussions of the work and instrumental help in editing a manuscript during the XXVIII International Workshop on "Geometry in Physics", held on 30.06.-07.07.2019 in Bia owieża, Poland. They also are cordially appreciated to Profs. Joel Lebowitz, Denis Blackmore and Nikolai N. Bogolubov (Jr.) for instructive discussions, useful comments and remarks on the work. A special author's appreciation belongs to Prof. Joel Lebowitz for the invitation to take part in the 121-st Statistical Mechanics Conference, held May 12-14, 2019 in the Rutgers University, New Brunswick, NJ, USA. Personal A.P.'s acknowledgement belongs to the Department of Physics, Mathematics and Computer Science of the Cracov University of Technology for a local research grant F-2/370/2018/DS.

References

1. Akhiezer AI, Peletminsky SV (2013) Methods of statistical physics. Pergamon Press.
2. Balescu R (1975) Equilibrium and Non-Equilibrium Statistical Mechanics, Wiley, New York.
3. Bazarov IP, Gevorkian EV, Nikolaev PN (1989) Nonequilibrium thermodynamics and physical kinetics. Moscow University Press.
4. Bogolubov NN (1960) Problems of dynamical theory in statistical physics. Geophysics Research Directorate, AF Cambridge Research Laboratories, Air Force Research Division, United States Air Force.
5. Bogolubov NN (1975) Microscopic solutions of the Boltzmann-Enskog equation in kinetic theory for elastic balls. Theor Math Phys 24: 804-807. [Link: https://bit.ly/3BCDowd](https://bit.ly/3BCDowd)
6. Bogolubov NN, Bogolubov NN (1994) Introduction to Quantum Statistical Mechanics. Gordon and Breach, New York, London.
7. Bogolubov NN, Sadovnikov BI (1975) Some problems of statistical mechanics. Vyshaya Shkola Publisher.
8. Bogolubov NN, Sadovnikov BI, Shumovsky AS (1984) Mathematical methods of statistical mechanics of model systems. CRC Press Publisher.
9. Bogolubov NN, Prykarpatsky AK, Samoilenko VH (1991) Hamiltonian structure of hydrodynamical Benney type equations and associated with them Boltzmann-Vlasove equations on axis. Preprint of the Institute of Mathematics of NAS of Ukraine, N91 25: 43.
10. Chapman S, Cowling T (1952) Mathematical theory of non-uniform gases. Published by Cambridge University Press, London and New York.
11. Libov R (1969) Introduction to the theory of kinetic equations, Wiley. [Link: https://bit.ly/3v8stYP](https://bit.ly/3v8stYP)
12. Petrina DYA, Gerasimenko VI, Malyshev PV (2002) Mathematical foundations of classical statistical mechanics. CRC Press Publisher. [Link: https://bit.ly/3BGt71Z](https://bit.ly/3BGt71Z)
13. Bogolubov NN, Brankov JG, Zagrebnov VA, Kurbatov AM, Tonchev NS (1981) Approximating Hamiltonian Method in Statistical Physics. Bulgarian Academy of Sciences Publ., Sophia.
14. Kozlov VV (2002) Thermal Equilibrium in the Sense of Gibbs and Poincare, Inst. Komp'yut. Issled. Publisher, Izhevsk.



15. Kozlov VV (2008) Gibbs Ensembles and Nonequilibrium Statistical Mechanics. Regulyarnaya i Khaoticheskaya Dinamika Publisher, Izhevsk.
16. The Vlasov kinetic equation (2011) dynamics of continuum and turbulence. Regular and Chaotic Dynamics 16: 602–622. [Link: https://bit.ly/2X88WuX](https://bit.ly/2X88WuX)
17. Lions PL, Perthame B (1991) Propagation of Moments and Regularity for the 3-Dimensional Vlasov-Poisson System. Invent Math 105: 415-430. [Link: https://bit.ly/3oYBd2E](https://bit.ly/3oYBd2E)
18. Marsden JE, Morrison PJ, Weinstein A (1984) The Hamiltonian structure of the BBGKY hierarchy equations. Contemp Math 28: 115-124. [Link: https://bit.ly/3oX04DH](https://bit.ly/3oX04DH)
19. Mikhaylov AI (2011) The functional mechanics: Evolution of the moments of distribution function and the Poincare recurrence theorem. Vestn Samar Gos Tekh Univ Fiz Mat Nauki 124–133. [Link: https://bit.ly/3ABCb6Q](https://bit.ly/3ABCb6Q)
20. Trushechkin AS (2014) Microscopic solutions of kinetic equations and the irreversibility problem. Proceedings of the Steklov Institute of Mathematics 285: 251–274. [Link: https://bit.ly/3FJCrEE](https://bit.ly/3FJCrEE)
21. Villani C (2002) A review of mathematical topics in collisional kinetic theory. Handbook of Mathematical Fluid Dynamics, edited by S. Friedlander and D. Serre, Elsevier Science Publisher. [Link: https://bit.ly/3oWhXTe](https://bit.ly/3oWhXTe)
22. Zubarev DN (1974) Nonequilibrium Statistical Thermodynamics. Consultants Bureau, NY. [Link: https://bit.ly/3awJJKu](https://bit.ly/3awJJKu)
23. Bogolubov NN, Prykarpatsky AK (1986) Quantum method of Bogolyubov generating functions in statistical physics: Lie current algebra, its representations and functional equations. Soviet Journal of Particles and Nuclei 17: 789-827. [Link: https://bit.ly/3mNdxv9](https://bit.ly/3mNdxv9)
24. Bogolubov NN, Prykarpatsky AK (1986) NN Bogolyubov's quantum method of generating functionals in statistical physics: the current Lie algebra, its representations and functional equations, Ukrainskii Matematicheskii Zhurnal 38: 284-289. [Link: https://bit.ly/3mLCirS](https://bit.ly/3mLCirS)
25. Daletsky YuL, Kadobyan RM (1994) The Poisson structures hierarchy and interacting systems dynamics. Proceed. Ukrainian Academy of Sciences 8: 21-26.
26. Ivankiv LI, Prykarpatski AK, Samulyak RV (1992) Nonequilibrium statistical mechanics of manyparticle systems in bounded domain with surface peculiarities and adsorption phenomenon. Preprint N1-92, Institute for applied Problems of Mechanics and Mathematics of NASU, Lviv, Ukraine.
27. Mandjavidze J, Sissakian A (2000) Generating functional method of N.N. Bogolubov and multiple production physics. [Link: https://bit.ly/3iY5GK7](https://bit.ly/3iY5GK7)
28. Mendes RV (2017) Current algebra, statistical mechanics and quantum models. [Link: https://bit.ly/3aChahK](https://bit.ly/3aChahK)
29. Menikoff R (1974) Generating functionals determining representation of a nonrelativistic local current algebra in the N/V -limit. J Math Phys 15: 1394-1408. [Link: https://bit.ly/3iY5Np1](https://bit.ly/3iY5Np1)
30. Menikoff R, Sharp D (1975) Representation of a local current algebra: their dynamical determination. J Math Phys 16: 2341-2352. [Link: https://bit.ly/3oVL210](https://bit.ly/3oVL210)
31. Volovich IV (2008) Time irreversibility problem and functional formulation of classical mechanics. Vestn Samar Gos Tekh Univ Estestvennonauchn.
32. Vlasov AA (1966) Statistical distribution functions. Moscow, Nauka Publisher. [Link: https://bit.ly/3mOMlwb](https://bit.ly/3mOMlwb)
33. Abraham R, Marsden J (2008) Foundations of Mechanics. Second Edition, Benjamin Cummings, NY.
34. Arnold VI (1978) Mathematical Methods of Classical Mechanics. Springer, NY.
35. Weinstein A (1983) Sophus Lie and symplectic geometry. Expos Math 1: 95–96.
36. Weinstein A (1983) The local structure of Poisson manifolds. J Differential Geom 18: 523–557. [Link: https://bit.ly/30cuyY2](https://bit.ly/30cuyY2)
37. Hentosh OY, Balinsky AA, Prykarpatski AK (2020) The generalized centrally extended Lie algebraic structures and related integrable heavenly type equations. Carpathian Math Publ 1: 242–264. [Link: https://bit.ly/3F11fwP](https://bit.ly/3F11fwP)
38. Bogolubov NN, Prykarpatsky AK, Samoilenko VH (1986) Functional equations of N.N. Bogolubov and associated with them symplectic Lie-Poisson-Vlasov structure. Ukr Math Journal 38: 747-778.
39. Prykarpatsky AK, Mykytyuk IV (1991) Algebraic aspects of integrability of nonlinear dynamical systems on manifolds. Kiev Nauk Dumka Publisher.
40. Blackmore D, Prykarpatsky AK, Samoilenko VH (2011) Nonlinear dynamical systems of mathematical physics: spectral and differential-geometrical integrability analysis. World Scientific Publ., NJ, USA.
41. Dirac PAM (1935) The Principles of Quantum Mechanics. Second edition. Oxford, Clarendon Press. [Link: https://go.nature.com/30b8Eo3](https://go.nature.com/30b8Eo3)
42. Kac M (1956) Some Stochastic Problems in Physics and Mathematics, Colloquium lectures in pure and applied science. Magnolia Petroleum Co 2. [Link: https://bit.ly/3BGsWUn](https://bit.ly/3BGsWUn)
43. Mikhaylov AI (2011) The functional mechanics: Evolution of the moments of distribution function and the Poincare recurrence theorem. p-Adic Numbers Ultrametric Anal Appl 3: 205-211. [Link: https://bit.ly/30cuql2](https://bit.ly/30cuql2)