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Short Communication

Boundary value problem for the third-order equation with multiple characteristics

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Annotation

The article constructs a unique solution to a tertiary-order equation with multiple characteristics with boundary conditions that include all possible local boundary conditions. The uniqueness of the solution of boundary value problems is proved by the method of integral equations using the sign-definiteness of quadratic forms. When proving the existence of a solution to the problem, Green's function method, the theory of integral equations and potentials are used.

Abbreviations

ARK: Abdukomil Risbekovich Khashimov; KDA: Khilola Djumaniyazova Atamuradovna

Introduction

In recent years, the desire for mathematical justification of natural phenomena is connected with the study of their mathematical models. For example, the compilation of mathematical models of wave propagation with large amplitudes, tsunami propagation in the oceans, nonlinear wave propagation in shallow waters, hydro-magnetic wave propagation in a cold plasma and acoustic waves in anharmonic crystals is connected with linear and nonlinear equations of odd order [1-6]. Therefore, the study of an odd-order equation with different boundary conditions is one of the urgent problems. Since the third-order equation describes the propagation of non-standard waves in various media, it is precisely here that a study is required to study the asymptotic properties of solutions to these equations. One of the methods of such research is the Saint-Venant principle, which occupies a special place in the theory of elasticity [7-12]. To investigate the asymptotic properties of solutions to boundary value problems using the Saint-Venant principle, it is necessary that a solution to this problem be constructed in a bounded domain. Therefore, the problem under consideration in the proposed article is relevant.

Formulation of the problem and main results

The purpose of this article is to study the equations

$$Lu \equiv \frac{\partial^3 u}{\partial x^3} - \frac{\partial^2 u}{\partial t^2} = 0, \tag{1}$$

In area $\Omega = \left\{ \left(\tilde{o}, t \right) : 0 < x < 1, 0 < t < T \right\}$. Equation (1) is one of the representatives of the third-order equation with multiple characteristics. It is known that the fundamental solutions of Eq. (1) have the following form [13].

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$$U\left(x-\xi;t-\tau\right)=\left|t-\tau\right|^{1/3} f\left(\frac{x-\xi}{\left|t-\tau\right|^{2/3}}\right),\ x\neq\xi,\ t\neq\tau; \tag{2}$$

$$V\left(x-\xi;t-\tau\right)=\left|t-\tau\right|^{1/3} \varphi\left(\frac{x-\xi}{\left|t-\tau\right|^{2/3}}\right),\ x<\xi,\ t\neq\tau. \tag{3}$$

Here

$$f(z)=\frac{2}{3}|z|^{1/2}\int_z^{\infty}\eta^{-3/2}f^*(\eta)\,d\eta+c^+,\ z>0,$$

$$f(z)=\frac{2}{3}|z|^{1/2}\int_{-\infty}^{z}\eta^{-3/2}f^*(\eta)\,d\eta+c^-,\ z<0,$$

$$\varphi(z)=\frac{2}{3}|z|^{1/2}\int_{-\infty}^{z}\eta^{-3/2}\varphi^*(\eta)\,d\eta+c,\ z<0,$$

$$f^*(z)=\int_0^{\infty}\exp\left(-\frac{\lambda^{3/2}}{\sqrt{2}}\right)\cos\left(\frac{\lambda^{3/2}}{\sqrt{2}}+\lambda z\right)d\lambda,\ -\infty<z<\infty,$$

$$\varphi^*(z)=\int_0^{\infty}\exp\left(\lambda z-\lambda^{3/2}\right)dz+\int_0^{\infty}\exp\left(-\frac{\lambda^{3/2}}{\sqrt{2}}\right)\sin\left(\frac{\lambda^{3/2}}{\sqrt{2}}+\lambda z\right)d\lambda,\ z<0,$$

$$z=(x-\xi)\left|t-\tau\right|^{-2/3}.$$

Here c^+, c^- – integral constants in the positive and negative parts of the axis z respectively.

Functions $f(z),\varphi(z)$, $f^*(z),\varphi^*(z)$ is called the Airy functions for which the relation

$$f''(z)+\frac{2}{3}zf^*(z)=0,\ \varphi'''(z)+\frac{2}{3}z\varphi^*(z)=0,$$

$$\int_{-\infty}^{\infty}f^*(z)=\pi,\quad \int_{-\infty}^{0}f^*(z)=\frac{2\pi}{3},\quad \int_0^{\infty}f^*(z)=\frac{\pi}{3},$$

$$\int_{-\infty}^{0}\varphi^*(z)=0.$$

In the domain Ω consider equation (1) with the boundary condition:

$$u(x,0)=\psi_1(x),\ u(x,1)=\psi_2(x),\ 0\le x\le 1, \tag{4}$$

$$\alpha_1(t)u(0,t)+\alpha_2(t)u_{xx}(0,t)=\varphi_1(t),$$

$$u_x(1,t)=\varphi_2(t),\ 0\le t\le T \tag{5}$$

$$\beta_1(t)u(1,t)+\beta_2(t)u_x(1,t)+\beta_3(t)u_{xx}(1,t)=\varphi_3(t)$$

$$0\le t\le T \tag{6}$$

We note that we considered similar problems in [14,15] only for third-order equations of a different form.

Theorem 1: Let the following conditions be satisfied

$$\alpha_2\neq 0,\ \beta_3\neq 0,\ 2\beta_1\beta_3-\beta_2^2\ge 0,\ \frac{\alpha_1}{\alpha_2}\le 0$$

Then problem (1),-(4) does not have more than one solution in the class $u\in C_{x,t}^{3,2}(\Omega)\cap C_{x,t}^{2,1}(\bar{\Omega})$

Let problems (1), (4)-(6) have two solutions $u_1(x,t)$, $u_2(x,t)$. Then assuming $v(x,t)=u_1(x,t)-u_2(x,t)$ we obtain the following homogeneous problem with respect to the function $v(x,t)$.

$$Lv\equiv\frac{\partial^3 v}{\partial x^3}-\frac{\partial^2 v}{\partial t^2}=0,\qquad (x,t)\in\Omega,$$

$$v(x,0)=0,\ v(x,1)=0,$$

$$\alpha_1(t)v(0,t)+\alpha_2(t)v_{xx}(0,t)=0,\ v_x(1,t)=0,$$

$$\beta_1(t)v(1,t)+\beta_2(t)v_x(1,t)+\beta_3(t)v_{xx}(1,t)=0.$$

Consider the identity

$$\int_0^1\int_0^1 L(v)v\,dxdt=0.$$

Integrating by parts, we get

$$-\int_0^1\int_0^1 v_t^2(x,t)\,dxdt-\int_0^1\left(\begin{array}{c}\frac{\beta_1}{\beta_3}v^2(1,t)+\frac{\beta_2}{\beta_3}v_x(1,t)v(1,t)\\ -\frac{\alpha_1(t)}{\alpha_2(t)}v^2(0,t)+\frac{1}{2}v_x^2(1,t)\end{array}\right)dt=0$$

Hence, by virtue of the conditions of the theorem, the quadratic form

$$Q(t)\equiv\frac{\beta_1}{\beta_3}v^2(1,t)+\frac{\beta_2}{\beta_3}v_x(1,t)v(1,t)+\frac{1}{2}v_x^2(1,t)$$

will be positive definite, i.e. quadratic form $Q(t)$ can be written in the following form:

$$\frac{\beta_1}{\beta_3}v^2(1,t)+\frac{\beta_2}{\beta_3}v_x(1,t)v(1,t)+\frac{1}{2}v_x^2(1,t)=\lambda_1 v_x^2+\lambda_2 v^2,$$

$$\lambda_1>0,\ \lambda_2>0.$$

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Here λ_1 and λ_2 – roots of the characteristic polynomial of a matrix of a quadratic form Q. Then, by virtue of the conditions of the theorem we have $v(x,t)=0$ in Ω and due to the continuity of the function $v(x,t)$ in $\overline{\Omega}$ we get that $v(x,t)=0$ in $\overline{\Omega}$.

Theorem 2: Let the conditions of Theorem 1 be satisfied and the following conditions

$$\psi_1(x),\psi_2(x)\in C^1([0,1]);\ \varphi_1(t)\in C^1([0,T]);$$
$$\varphi_2(t),\varphi_3(t)\in C([0,T])$$

Then problem (1) (4),-(6) has a unique solution in class $u\in C^{3,2}_{x,t}(\Omega)\cap C^{2,1}_{x,t}(\overline{\Omega})$.

The proof of Theorem 2, i.e. the existence theorem, the solution of problem (1), (4)-(6) is carried out by Green's function method using the theory of potentials of fundamental solutions of equation (1) and on the basis of the theory of the system of Fredholm integral equations of the second kind.

Conclusions

Based on the research, the following conclusions can be drawn:

- In the proposed method for studying boundary value problems, we will apply all possible local boundary value problems for equations (1), since the boundary condition (5), (6) includes all possible boundary conditions;

- Based on the results of this work, one can study the asymptotic properties of solutions to equation (1) in the neighborhood of infinitely distant points of the boundary and irregular points of the boundary;

- Based on the results of this work, it is now possible to construct a solution to boundary value problems in unbounded domains.

Author contributions

Conceptualization, methodology, validation, formal analysis, investigation A.R.K.; validation, formal analysis, K.D.A. All authors have read and agreed to the published version of the manuscript.

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