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Research Article

Spectral analysis of the Sturm-Liouville operator given on a system of segments

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The spectral analysis of the Sturm-Liouville operator defined on a finite segment is the subject of an extensive literature [1,2]. Sturm-Liouville operators on a finite segment are well studied and have numerous applications [1-6]. The study of such operators already given on the system segments (graphs) was received in the works [7,8]. This work is devoted to the study of operators

$$(L_q y)(x) = col[-y_1''(x) + q_1(x)y_1(x), -y_2''(x) + q_2(x)y_2(x)],$$

where $y(x) = col[y_1(x), y_2(x)] \in L^2(-a,0) \oplus L^2(0,b) = H, q_1(x), q_2(x) -$ real function $q_1 \in L^2(-a,0), q_2 \in L^2(0,b).$ Domain of definition L_q has the form

$$\vartheta(L_q) = y = (y_1, y_2) \in H;\ y_1 \in W_1^2(-a,0),\ y_2 \in W_2^2(0,b),\ y_1'(-a) = 0,\ y_2'(b) = 0;\ y_2(0) + py_1'(0) = 0;\ y_1(0) + py_2'(0) = 0$$

$(p \in \mathbb{R}, p \neq 0)$. Such an operator is self-adjoint in H. The work uses the methods described in work [9,10]. The main result is as follows: if the q_1, q_2 are small (the degree of their smallness is determined by the parameters of the boundary conditions and the numbers a,b),), then the eigenvalues $\{\lambda_k(0)\}$ of the unperturbed operator L_o are simple, and the eigenvalues $\{\lambda_k(q)\}$ of the perturbed operator L_q are also simple and located small in the vicinity of the points $\{\lambda_k(0)\}$.

Introduction

The operator L_q describes the oscillatory processes of a system located on two intervals. In other words, the vibrations of connected rods are connected with the spectral analysis of the operator L_q. The purpose of this work is to establish at what smallness of the potentials the spectrum of the problem differs slightly from the spectrum of the unperturbed problem.

Unperturbed operator

Consider the Hilbert space $H = L^2(-a,0) \oplus L^2(0,b)$, $(a,b \geq 0)$ by vector functions $y(x) = col[y_1(x), y_2(x)]$, where $y_1 \in L^2(-a,0), y_2 \in L^2(0,b)$. Define in H a linear operator

$$\left(L_q y\right)(x) = col\left[-y_1''(x) + q_1(x)y_1(x), -y_2''(x) + q_2(x)y_2(x)\right], \tag{1.1}$$

Where q_1, q_2 - real function and $q_1(x) \in L^2(-a,0), q_2 \in L^2(0,b)$.

The domain of definition of the operator L_q has the form,

$$\vartheta(L_q) = \{y = col[y1, y2] \in H;\ y_1 \in W_1^2(-a,0), y_2 \in W_2^2(0,b),\ y_1'(-a) = 0,\ y_2'(b) = 0,\ y_2(0) + py_1(0) = 0,$$
$$y_1'(0) + py_2'(0) = 0\} \tag{1.2}$$

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$(p \in \mathbb{R}, p \neq 0)$. The operator L_q (1.1),(1.2) is symmetric because

$$< L_q y, g > - < y, L_q g > = -y_1'(x)\bar{g}_1(x)\Big|_{-a}^{0} + y_1(x)\bar{g}_1'(x)\Big|_{-a}^{0} - y_2'(x)\bar{g}_2(x)\Big|_{0}^{b} + y_2(x)\bar{g}_2'(x)\Big|_{0}^{b} =$$

$$= py_2'(0)\bar{g}_1(0) - p\bar{g}_2'(0)y_1(0) - p\bar{g}_1(0)y_2'(0) + py_1(0)\bar{g}_2'(0) = 0.$$

It is easy to show that L_q (1.1),(1.2) is self-adjoint.

Primary study the unperturbed operator L_0 $(q_1=q_2=0)$. Respectively, the function of operator L_0 is a solution to the equations

$$-y_1'' = \lambda^2 y_1, -y_2'' = \lambda^2 y_2 \quad (\lambda \in \mathbb{C}) \tag{1.3}$$

and satisfies the boundary conditions (1.2). From the first two boundary conditions we find that

$$y_1 = A\cos\lambda(x+a), y_2 = B\cos\lambda(x-b), \tag{1.4}$$

where $A, B, C \in \mathbb{C}$. Second and third boundary $\{y_1, y_2\}$ (1.4) give a system of equations for A and B,

$$\begin{cases} Ap\cos\lambda a + B\cos\lambda b = 0, \\ -A\lambda\sin\lambda a + Bp\lambda\sin\lambda b = 0. \end{cases} \tag{1.5}$$

This system has non-trivial solution A and B, if only its determinant $\Delta(0,\lambda)=0$, where

$$\Delta(0,\lambda) = \lambda p^2\cos\lambda a\sin\lambda b + \lambda\cos\lambda b\sin\lambda a. \tag{1.6}$$

If $\lambda = 0$, then $y_1 = A$, $y_2 = B$ and from $y_2(0) + py_1(0) = 0$ follows that $B = -pA$. So $y = A col[1, -p]\,(A \in \mathbb{C})$ operator's own function L_0, responding to its own value $\lambda=0$ With $\lambda \neq 0$ from $\Delta(0,\lambda)=0$ follows

$$p^2\cos\lambda a\sin\lambda b + \cos\lambda b\sin\lambda a = 0, \tag{1.7}$$

Remark 1

If $p = \pm 1$, *that from (1.7) follow* $\sin\lambda(a+b) = 0$ *and hence the own numbers have the form*

$$\lambda_n = \frac{\pi n}{a+b} \quad (n \in \mathbb{Z}),\ a+b \neq 0 \tag{1.8}$$

Consider the general case, not assuming, that $p \pm 1$ and write equality (1.7) in form

$$(p^2+1)\sin\lambda(a+b) - (1-p^2)\sin\lambda(b-a) = 0$$

or

$$\sin\lambda(a+b) - \frac{(1-p^2)}{(1+p^2)}\sin\lambda(b-a) = 0 \tag{1.9}$$

Let be

$$\lambda(b+a) = w, \frac{1-p^2}{1+p^2} = k, \frac{b-a}{b+a} = q, \tag{1.10}$$

then it is obvious that $|k| \le 1, |q| \le 1$, and the equation (1.9) has form

$$f(w) = 0;\ f(w) \overset{\text{def}}{=} \sin w - k\sin qw \tag{1.11}$$

The function $f(w)$ is odd, therefore it is enough to find its zeros $f(w)$ on the ray $\mathbb{R}_+$.

Show that the zeros of $f(w)$ are simple. Assuming the opposite, suppose that w - repeated root, then from $f(w)$ and $f'(w)=0$, follows, that

$$\begin{cases} \sin w = k\sin qw \\ \cos w = kq\cos qw \end{cases}$$

That means

$$k^2\sin^2 qw + k^2q^2\cos^2 qw = 1$$

that's why

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$$\sin^2 qw + q^2\cos^2 qw = \frac{1}{k^2} \qquad (1.12)$$

Since $|k| \le 1$ ($k = 1$ which p=0,isimpossiblebyassumption) and $|q| < 1$, then from (1.12) follows that the left side $\sin^2 qw + q^2\cos^2 qw \le 1$, and right side $\frac{1}{k^2} > 1$. That's why roots *f(w)* are simple.

Theorem 1

Roots $\{\lambda_s(0)\}$ of the characteristic function $\Delta(0,\lambda)$ (1.6) are simple except $\lambda_0(0)=0$ which is duble multiple and they have the form,

$$\Lambda_0 = \{0, \lambda_s(0) = \pm\frac{w_s}{a+b}, w_s > 0; \sin w_s = k\sin qw_s\}, \qquad (1.13)$$

where k i q-have of form (1.10), and the numbers $w_s \epsilon \mathbb{R}_+$ are numbered in ascending order.

Remark 2

Greatest positive root w_1 equation f(w)=0 obviously lies in the interval $\frac{\pi}{2} < w_1 < \pi$ *and that mean* $\frac{\pi}{2(a+b)} < \lambda_1(0) < \frac{\pi}{a+b}$.

Eigenfunctions $\varphi(0,\lambda_s(0))$ of operator $L_{0,}$ responding $\lambda_s(0)\epsilon\Lambda_0$ (1.13) are equal

$$\varphi(0,\lambda_s(0)) = A_s col\left[\cos\lambda_s(0)b\cos\lambda_s(0)(x+a), -p\cos\lambda_s(0)a\cos\lambda_s(0)(x-b)\right], \qquad (1.14)$$

which is an obvious consequence (1.4), (1.5)

Perturbed operator

Let's move on to the perturbed operator L_q. The equation for the eigenfunction $y = col[y_1, y_2]$ of operator L_q has the form

$$-y_1'' + q_1 y_1 = \lambda^2 y_1, -y_2'' + q_2 y_2 = \lambda^2 y_2 \qquad (2.1)$$

Consider the integral equations

$$\begin{cases} y_1(x) = A\cos\lambda(x+a) + \int_a^x \frac{\sin\lambda(x-t)}{\lambda} q_1(t) y_1(t)dt; \\ y_2(x) = B\cos\lambda(x-b) - \int_x^b \frac{\sin\lambda(x-t)}{\lambda} q_2(t) y_2(t)dt. \end{cases} \qquad (2.2)$$

Then $\{y_k(x)\}-$) solution (2.2) satisfy the equations (2.1), and the first boundary conditions (1.2) correspond to y_1, y_2.

Solvability of the integral equation (2.2) for y_1. Definition of Volterra operator in $L^2(-a,0)$,

$$(K_1 f)(x) = \int_{-a}^x K_1(x,t) q_1(t) f(t)dt \quad (f \in L^2(-a,0)), \qquad (2.3)$$

where

$$(K_1 f)(x) = \frac{\sin\lambda(x-t)}{\lambda}. \qquad (2.4)$$

Then the first of the equations in (2.2) will take the form

$$(I - K_1)y_1 = A\cos\lambda(x-t), \qquad (2.5)$$

And that means

$$y_1 = \sum_{n=0}^{\infty} K_1^n A\cos\lambda(x+a) \qquad (2.6)$$

where

$$(K_1^n f)(x) = \int_{-a}^x K_{1,n}(x,t) q_1(t) f(t)dt, \qquad (2.7)$$

For cores $K_{1,n}(x,t)$ the recurrence relations are valid

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$$K_{1,n+1}(x,t)=\int_t^x K_1(x,s)K_{1,n}(s,t)q_1(s)dt \quad (n>1) \tag{2.8}$$

where $K_1(x,t)$ have from (2.4)

We need kernel estimates $K_{1,n}(x,t)$ to prove the solvability of the integral equations.

Lemma 1

The kernels $K_{1,n}(x,t)$ (2.8) satisfy the inequalities

$$|K_{1,n}(x,t)|\le ch\beta(x-t)\frac{(x-t)^n}{n^n}\cdot\frac{\sigma_1^{n-1}(x)}{(n-1)!}, \tag{2.9}$$

where

$$\beta=Im\lambda,\quad \sigma_1(x)=\int_{-a}^{x}|q_1(t)|dt. \tag{2.10}$$

The proof of the estimates (2.9) is carried out by induction From (2.6) it follows, that

$$y_1(\lambda,x)=A\cos\lambda(x+a)+A\int_{-a}^{x}\sum_{n=1}^{\infty}K_{1,n}(x,t)q_1(t)\cos\lambda(t+a)dt=A\cos\lambda(x+a)+A\int_{-a}^{x}N_1(x,t,\lambda)q_1(t)\cos\lambda(t+a)dt,$$

where

$$N_1(x,t,\lambda)=\sum_{n=1}^{\infty}K_{1,n}(x,t).$$

it follows from the estimates (2.9) that this series converges and

$$|N_1(x,t,\lambda)|\le\cosh\beta(x-t)(x-t)\exp\left[(x-t)\sigma_1(x)\right]$$

Similar reasoning is valid for the second equation (2.2).

Theorem 2

Integral equations (2.2) are resolved and,-

$$\begin{cases} y_1(\lambda,x)=A\left(\cos\lambda(x+a)+\int_{-a}^{x}N_1(x,t,\lambda)q_1(t)\cos\lambda(t+a)dt\right); \\ y_2(\lambda,x)=B\left(\cos\lambda(b-x)-\int_{x}^{b}N_2(x,t,\lambda)q_2(t)\cos\lambda(b-t)dt\right), \end{cases} \tag{2.11}$$

In this case, the kernels $\{N_k(x,t,\lambda)\}$) satisfy the estimates

$$|N_k(x,t,\lambda)|\le\cosh\beta(x-t)\cdot(x-t)\cdot exp\{(x-t)\sigma_k(t)\}\ (k=1,2), \tag{2.12}$$

where

$$\beta=Im\lambda,\quad \sigma_1(x)=\int_{-a}^{x}|q_1(t)|dt,\quad \sigma_2(x)=\int_{x}^{b}|q_2(t)|dt. \tag{2.13}$$

To find a characteristic function $\Delta(q,\lambda)$ the operator L_q L uses the last boundary conditions (1.2) for the $\{y_k(\lambda,x)\}$,), as a result, we obtain a one-row system of equations for A and B,-

$$\begin{cases} pA\left(\cos\lambda a+\int_{-a}^{0}N_1(0,t,\lambda)q_1(t)\cos\lambda(t+a)dt\right)+B\left(\cos\lambda b-\int_{0}^{b}N_2(0,t,\lambda)q_2(t)\cos\lambda(b-t)dt\right)=0, \\ A\left(-\lambda\sin\lambda a+\int_{-a}^{0}N_1'(0,t,\lambda)q_1(t)\cos\lambda(t+a)dt\right)+pB\left(\lambda\sin\lambda b-\int_{0}^{b}N_2'(0,t,\lambda)q_2(t)\cos\lambda(b-t)dt\right)=0 \end{cases} \tag{2.14}$$

System (2.14) at this value $q_1=q_2=o$ coincides with the system (1.5) and it has a nontrivial solution A, B, if its determinant $\Delta(q,\lambda)=0$, where

$$\Delta(q,\lambda)\overset{\text{def}}{=}\begin{vmatrix} p\left(\cos\lambda a+\int_{-a}^{0}N_1(0,t,\lambda)q_1(t)\cos\lambda(t+a)dt\right) & \cos\lambda b-\int_{0}^{b}N_2(0,t,\lambda)q_2(t)\cos\lambda(b-t)dt \\ -\lambda\sin\lambda a+\int_{-a}^{0}N_1'(0,t,\lambda)q_1(t)\cos\lambda(t+a)dt & p\left(\lambda\sin\lambda b-\int_{0}^{b}N_2'(0,t,\lambda)q_2(t)\cos\lambda(b-t)dt\right) \end{vmatrix} \tag{2.15}$$

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It follows that,

$$\Delta(q,\lambda)=\Delta(0,\lambda)+\Phi(\lambda), \tag{2.16}$$

where $\Delta(o,\lambda)$) have form (1.6), and $\Phi(\lambda)$ is equal

$$\begin{aligned}\Phi(\lambda)=p^2\{\lambda\sin\lambda b\int_{-a}^{0}N_1(0,t,\lambda)q_1(t)\cos\lambda(t+a)dt-\cos\lambda a\int_0^b N_2'(0,t,\lambda)q_2(t)\cos\lambda(b-t)dt-\\-\int_{-a}^{0}N_1(0,t,\lambda)q_1(t)\cos\lambda(t+a)dt\times\int_0^b N_2'(0,t,\lambda)q_2(t)\cos\lambda(b-t)dt\}-\\-\lambda\sin\lambda a\int_0^b N_2(0,t,\lambda)q_2(t)\cos\lambda(b-t)dt-\cos\lambda b\int_{-a}^{0}N_1'(0,t,\lambda)q_1(t)\cos\lambda(t+a)dt+\\+\int_0^b N_2(0,t,\lambda)q_2(t)\cos\lambda(b-t)dt\cdot\int_{-a}^{0}N_1'(0,t,\lambda)q_1(t)\cos\lambda(t+a)dt\end{aligned} \tag{2.17}$$

Let us formulate a theorem that shows how strongly the characteristic functions of the perturbed and unperturbed operators differ.

Theorem 3

Operator characteristic function $\Delta(q,\lambda)$ *(2.15) is expressed in terms of the operator* L_q *(1.1),(1.2) characteristic function* $\Delta(0,\lambda)$ *(1.6)* L_o *(*$q_1=q_2=o$*)) by the formula (2.16), where* $\Phi(\lambda)$ *has the fo (2.17) and is an entire function of exponential type while it satisfies the estimate*

$$|\Phi(\lambda)|\le ch\beta a\cdot ch\beta b\cdot(\delta_1|\lambda|+\delta_2), \tag{2.18}$$

where

$$\delta_1\overset{\text{def}}{=}\sigma_1 ae^{\sigma_1 a}+\sigma_2 be^{\sigma_2 b},\ \delta_2\overset{\text{def}}{=}\sigma_1 e^{\sigma_1 a}+\sigma_2 e^{\sigma_2 b}+\sigma_1\sigma_2(a+b)e^{\sigma_1 a+\sigma_2 b} \tag{2.19}$$

and $\beta=Im\lambda,\sigma_1=\sigma_1(0),\sigma_2=\sigma_2(0)$.

Proof The estimates are similarly (2.12) valid

$$\left|\frac{\partial}{\partial x}N_k(x,t,\lambda)\right|\le ch\beta(x-t)exp\{\sigma_k(x)(x-t)\}\ (k=1,2),$$

therefore, it follows from (2.17) that

$$|\Phi(\lambda)|\le p^2\{|\lambda|ch\beta b\cdot\cos\beta a\cdot e^{\sigma_1 a}\sigma_1 a+ch\beta a\cdot ch\beta b\cdot e^{\sigma_2 b}\sigma_2+a\cdot ch\beta a\cdot ch\beta b\cdot e^{\sigma_1 a}\cdot e^{\sigma_2 b}\sigma_1\sigma_2\}+$$

$$+|\lambda|ch\beta a ch\beta b e^{\sigma_2 b}\sigma_2 b+ch\beta b\cdot ch\beta a\cdot e^{\sigma_1 a}\sigma_1+bch\beta a\cdot ch\beta b\cdot e^{\sigma_1 a}\cdot e^{\sigma_2 b}\sigma_1\sigma_2.$$

Thus,

$$|\Phi(\lambda)|\le ch\beta b\cdot ch\beta a\{\sigma_1\cdot e^{\sigma_1 a}(1+|\lambda|p^2 a)+\sigma_2\cdot e^{\sigma_2 b}(b|\lambda|+p^2)+\sigma_1\sigma_2 e^{\sigma_1 a+\sigma_2 b}(b+p^2 a)\}$$

And since $p^2<1$,) then

$$|\Phi(\lambda)|\le ch\beta a\cdot ch\beta b\{|\lambda|(\sigma_1 ae^{\sigma_1 a}+b\sigma_2 e^{\sigma_2 b})+\sigma_1 e^{\sigma_1 a}+\sigma_2 e^{\sigma_2 b}+\sigma_1\sigma_2 e^{\sigma_1 a+\sigma_2 b}(b+a)\}$$

which proves (2.18). ■

Basic assessments

Characteristic function $\Delta(o,\lambda)$ (1.6) taking into account these (1.7),(1.8) is equal to,-

$$\Delta(0,\lambda)=\lambda(p^2+1)Q(\lambda);\ Q(\lambda)\overset{\text{def}}{=}\sin\lambda(a+b)-k\sin q\lambda(a+b), \tag{3.1}$$

where q,k has form (1.10) and $|k|\le 1,|q|\le 1$. Let us expand $Q(\lambda)$by the Taylor formula in a real neighborhood of the point $\lambda_s(0)(\ne 0)$ (1.13),

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$$Q(\lambda)=(\lambda-\lambda_s)Q'(\lambda_s)+\frac{(\lambda-\lambda_s)^2}{2}Q''(\xi_s)=(\lambda-\lambda_s)Q'(\lambda_s)\left(1+\frac{(\lambda-\lambda_s)^2}{2}\cdot\frac{Q''(\xi_s)}{Q'(\lambda_s)}\right),$$

where $\lambda\in\mathbb{R}$ i $\xi_s=\lambda_s+\theta(\lambda-\lambda_s)\,(|\theta|\le 1)$ for all λ satisfy the condition

$$|\lambda-\lambda_s|<\left|\frac{Q'(\lambda_s)}{Q''(\xi_s}\right| \quad (3.2)$$

the inequality is true

$$Q(\lambda)>\frac{|\lambda-\lambda_s|}{2}Q'(\lambda_s) \quad (3.3)$$

Because

$$Q'(\lambda)=(a+b)[\cos\lambda(a+b)-kq\cos q\lambda(a+b)]$$

$$Q''(\lambda)=-(a+b)^2[\sin\lambda(a+b)-kq^2\sin\lambda q(a+b)] \quad (3.4)$$

then

$$|Q''(\lambda)|\le(a+b)^2(1+|kq^2|)<(a+b)^2(1+|k|) \quad (3.5)$$

To get a lower estimate for the $|Q'(\lambda_s)|$ we use the (3.4), then we get

$$(Q'(w))^2=(a+b)^2\left\{\cos^2 w-2kq\cos w\cdot\cos qw+k^2q^2\cos 2qw\right\}=(a+b)^2\cdot$$

$$\cdot\left\{1-\sin^2 w+k^2q^2(1-\sin^2 qw)-2kq\cos qw\cos w\right\},$$

Where *w=λ(a+b)* and *sin w=ksinqw*. This implies that

$$\begin{aligned}&(Q'(w))^2\ge(a+b)^2\left\{1+k^2q^2-\sin^2 w(1+q^2)-2|kq|\sqrt{(1-\sin^2 w)(1-\sin^2 qw)}\right\}\ge\\&\ge(a+b)^2\left\{1+k^2q^2-\sin^2 w(1+q^2)-2|kq|(1-k^2\sin^2 qw)\right\}\ge(a+b)^2\left\{1-|kq|^2-\sin^2 w(1-|q|^2)\right\}\ge\\&\ge(a+b)^2(|q|(1-|k|))(2-|q|-|qk|)>2(a+b)^2|q|(1-|q|)(1-|k|).\end{aligned} \quad (3.6)$$

Then

$$\begin{aligned}&|Q'(\lambda_s(0))|>\sqrt{2}(a+b)\sqrt{|q|(1-|k|)(1-|q|)}>\\&>(a+b)|q|(1-|q|)(1-|k|)=(a+b)|q|r,\end{aligned} \quad (3.7)$$

where

$$r=(1-|q|)(1-|k|)=4\frac{min(a,b)min(1,p^2)}{(a+b)(p^2+1)}<1, \quad (3.8)$$

Based on (1.10) therefore, according to (3.7), (3.8) the inequality (3.2) is certainly satisfied if

$$|\lambda-\lambda_s|<\frac{|q|r}{(a+b)(1+|k|)}$$

Lemma 2

For all real λ, from the neighborhood

$$|\lambda-\lambda_s|<\frac{|q|r}{(a+b)(1+|k|)}=R \quad (3.9)$$

of the zero $_s(0)$ of the function $\Delta(o,\lambda)$ (1.6), the inequality is valid

$$|\Delta(0,\lambda)|>\frac{|\lambda-\lambda_s(0)|}{2}|\lambda|(1+p^2)Q'(\lambda_s(0))>\frac{|\lambda-\lambda_s(0)|}{2}|\lambda|(1+p^2)(a+b)|q|r, \quad (3.10)$$

Where *r,q* has form (1.10), (3.8)

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It follows from the (2.16) that

$$|\Delta(q,\lambda)|>|\Delta(0,\lambda)|-|\Phi(\lambda)|.$$

We choose $\lambda \in R$ from the neighborhood (3.9) $|\lambda-\lambda_s(0)|<R$ of the zero $\lambda_s(0)(\neq 0)$ of the function $\Delta(0,\lambda)$, then using (2.18) ($\beta = 0$) and (3.10) we obtain that

$$|\Delta(q,\lambda)|>\frac{|\lambda-\lambda_s(0)|}{2}|\lambda|(1+p^2)Q'(\lambda_s(0))-\delta_1|\lambda|-\delta_2=|\lambda|\left(\frac{|\lambda-\lambda_s(0)|}{2}(1+p^2)Q'(\lambda_s(0))-\delta_1-\frac{\delta_2}{|\lambda|}\right),$$

where numbers δ_s - has form (2.19). Therefore $|\lambda-\lambda_s|<R$ (3.9), then

$$|\lambda|>|\lambda_s|-R>|\lambda_1|-R>\frac{\pi}{2(a+b)}-\frac{|q|r}{(a+b)(1+|k|)}>\frac{1}{a+b}\left(\frac{\pi}{2}-r\right)>0$$

based on remark 2 , and that mean

$$|\Delta(q,\lambda)|>|\lambda|\left(\frac{|\lambda-\lambda_s(0)|}{2}(1+p^2)Q'(\lambda_s(0))-\delta_1-\frac{\delta_2(a+b)}{\frac{\pi}{2}-r}\right)$$

if the first part of this inequality is greater than zero, then

$$|\lambda-\lambda_s(0)|>\frac{2\delta_1+\frac{4\delta_2(a+b)}{\pi-2r}}{(1+p^2)Q'(\lambda_s(0))}$$

then for such $\lambda \in R$ function $|\Delta(q,\lambda)|$ does not turn to zero. So, if

$$\frac{2\delta_1+\frac{4\delta_2(a+b)}{\pi-2r}}{(1+p^2)Q'(\lambda_s(0))}<|\lambda-\lambda_s(0)|<R, \quad (3.11)$$

then $|\Delta(q,\lambda)|\neq 0$ multiplicity (3.11)isn`t empty, if

$$\frac{2\delta_1+\frac{4\delta_2(a+b)}{\pi-2r}}{(1+p^2)Q'(\lambda_s(0))}<R,$$

and using (3.7) i (3.9), we find that this inequality will certainly be satisfied if

$$2\delta_1+\frac{4\delta_2(a+b)}{\pi-2r}<(1+p^2)\frac{q^2r^2}{1+|k|} \tag{3.12}$$

So if the δ_1 and δ_2 (2.19) are such that holds (3.12), then the function $\Delta(q,\lambda)$ on the multiplicity (3.11) does not turn to 0. The signs $\Delta(q,\lambda)$ and $\Delta(0,\lambda)$ on the left and right sides of multiplicity (3.11) coincide, and given that the signs of the function $\Delta(0,\lambda)$ on these parts are different, it follows that $\Delta(q,\lambda)$ it has at least one root on the multiplicity.

$$|\lambda-\lambda_s(0)|<\frac{2\delta_1+\frac{4\delta_2(a+b)}{\pi-2r}}{(1+p^2)Q'(\lambda_s(0))}$$

Lemma 3

If numbers δ_1 and δ_2 (2.19) satisfy inequality (3.12), where p,q,r has form (1.10) and (3.8), then in the surrounding area

$$|\lambda-\lambda_s(0)|<\frac{2\delta_1+\frac{4\delta_2(a+b)}{\pi-2r}}{(1+p^2)(a+b)|q|r} \tag{3.13}$$

the zeros $\lambda_s(0)$ of the function $\Delta(0,\lambda)$ (1.6) contains at least one root $\lambda_s(q)$, of the perturbed characteristic function $\Delta(q,\lambda)$ (2.19).

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Main result

To prove that the characteristic function $\Delta(q,\lambda)$ has no other zeros, except $\lambda_s(q)$ we use Rousche's theorem. Let us denote by γ_1 the contour in the $\mathbb{C}$, formed by the straight lines that connect the points $\pi\frac{l}{a+b}(1+i), \pi\frac{l}{a+b}(-1+i), \pi\frac{l}{a+b}(-1-i), \pi\frac{l}{a+b}(1-i), (l\in\mathbb{N})$. We need a lower estimate for the function $\Delta(o,\lambda)$) on the contour γ_1 or, taking into account (3.1) a lower estimate for the function $Q(\lambda)$. For $\lambda=\alpha+i\beta\in\mathbb{C}\ (c=a+b)$ have

$$Q(\lambda)=\sin(\alpha+i\beta)c-k\sin q(\alpha+i\beta)c=\sin\alpha c\cosh\beta c+i\cos\alpha c\sinh\beta c-k(\sin\alpha qc\cosh\beta qc+i\cos\alpha qc\sinh\beta qc),$$

then

$$|Q(\lambda)|^2=\sin^2\alpha c\cosh^2\beta c+k^2\sin^2\alpha qc\cosh^2\beta qc-2k\sin\alpha c\sin\alpha qc\cosh\beta qc\cosh\beta c+\cos^2\alpha c\sinh^2\beta c+$$

$$+k^2\cos^2\alpha qc\sinh^2\beta qc-2k\cos\alpha c\cos\alpha qc\sinh\beta c\sinh\beta qc=\cosh^2\beta c-\cos^2\alpha c+k^2(\cosh^2\beta qc-\cos^2\alpha qc)-$$

$$-2k\sin\alpha c\sin\alpha qc\cosh\beta qc\cosh\beta c-2k\cos\alpha c\cos\alpha qc\sinh\beta c\sinh\beta qc\geq(\cosh\beta c-|k|\cosh\beta qc)^2-$$

$$-(\cos^2\alpha c+k^2\cos^2\alpha qc)(1+|\sinh\beta c||\sinh\beta qc|)\geq(\cosh\beta c-|k|\cosh\beta qc)^2-(1+k^2)(1+|\sinh\beta c||\sinh\beta qc|).$$

It follows that

$$|Q(\lambda)|\geq(\cosh\beta c-|k|\cosh\beta qc)\sqrt{1-(1+k^2)\frac{(1+|\sinh\beta c||\sinh\beta qc|)}{(\cosh\beta c-|k|\cosh\beta qc)^2}}$$

Hence follows the statement

Lemma 4

At $\lambda=\alpha+i\beta\in\mathbb{C}$ for function $\Delta(o,\lambda)$ (3.1) the inequality is true

$$|\Delta(0,\lambda)|>|\lambda|(p+1)\cosh\beta q(a+b)\sqrt{1+k^2}\cdot$$
$$\cdot\left(1-|\sin\alpha(a+b)\sin\alpha q(a+b)|-\left(\cos^2\alpha(a+b)+k^2\cos^2\alpha q(a+b)\right)\frac{1+\cosh^2\beta(a+b)}{\cosh^2\beta q(a+b)(1+k^2)}\right)^{1/2} \quad (4.1)$$

Through γ_1 we denote the contour in $\mathbb{C}$ formed by the square with the vertices at the points

$\pi\frac{l}{a+b}(1+i), \pi\frac{l}{a+b}(-1+i), \pi\frac{l}{a+b}(-1-i), \pi\frac{l}{a+b}(1-i), (l\in\mathbb{N})$. On the vertical section (4.1) $\lambda=\frac{\pi l}{a+b}(1+\beta i)\ (-1<\beta<1)$ it follow that

$$|\Delta(0,\lambda)|>\frac{\pi l}{a+b}\sqrt{1+\beta^2}\,|p+1|\sqrt{1+k^2}\cosh\beta q(a+b)\left(1+\frac{1+\cosh^2\beta(a+b)}{\cosh^2\beta q(a+b)}\right)^{1/2},$$

and from theorem 3 it follows that for such λ we have

$$|\Phi(\lambda)|<\cosh\beta a\cosh\beta b(\delta_1|\lambda|+\delta_2),$$

then at $l\gg 1$ for $\forall\lambda=\frac{\pi l}{a+b}(1+\beta i)\ \beta\in[-1,1]$ we have

$$|\Delta(0,\lambda)|>|\Phi(\lambda)| \quad (4.2)$$

It is proved in a similar way that on the sides of the square γ_1 at $l\gg 1$ the inequality is true (4.2).

The following theorem can be formulated from the above.

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Theorem 4

Suppose that the functions $q_1(x)$ and $q_2(x)$ in (1.1) are such that inequality (3.12) holds, where p,q,r are of the form (1.10) and (3.8). Then in each neighborhood (3.13) of the zero $\lambda_s(0)$ of the characteristic function $\Delta(0,\lambda)$ (1.6) of the unperturbed operator L_o there is only one zero $\lambda_s(q)$ of the perturbed characteristic function $\Delta(q,\lambda)$ (2.19) of the operator L_q.

Therefore, when the potentials are small $q_1(x)$ and $q_2(x)$ which are expressed only in terms of the parameters of the boundary conditions (1.2) each corresponding value of the operator L_q is located in a small neighborhood of the corresponding value of the unperturbed value of the operator $L_{o.}$

Concluding remarks

Thus, we have shown that if the potentials are small, (3.12) holds, then the spectrum of the perturbed problem $|q_1(x)|+|q_2(x)|\neq 0$ differs little from the unperturbed problem. Consequently, the perturbed oscillations will be close to the unperturbed ones.

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