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Research Article

Some fixed point results in rectangular metric spaces

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Abstract

After motivation from Geraghty-type contractions and of Farhan, et al. we define α-admissible mappings and demonstrate the fixed point theorems for the above-mentioned contractions in rectangular metric space in this study. In the end, we discuss some consequences of our results as corollaries.

2010 MSC: 47H10, 54H25.

Introduction

Banach provided a method to find the fixed point in the entire metric space in 1922. Since then, numerous researchers have attempted to generalise this idea by working on the Banach fixed point theorem (see [1-9], [11-22],[26,27]). The term " α admissible mappings in metric space" pertains to the innovative concepts in mappings that Samet, et al. [27] pioneered in 2012. Recently, in 2013 Farhan, et al. [2] gave new contractions using α -admissible mapping in metric spaces. In continuation of generalization of Banach contraction principle, in 2018, Karapinar introduced the notion of interpolative contraction via revisiting Kannan contraction which involves exponential factors. Combining the interpolative contractions with linear and rational terms several authors defined hybrid contractions and proved fixed point theorems for these contractions see(16,24-25). We'll generalize Farhan's, et al. [2] contractions in the following paper and provide fixed point theorems for them.

Preliminaries

To prove our main results we need some basic definitions from literature as follows:

Definition 2.1: [10] Let $\aleph$ be a set. A rectangular metric space (RMS) is an ordered pair $(\aleph,\Omega)$ where Ω is a function $\Omega:\aleph\times\aleph\to\mathbb{R}$ such that

1. $\Omega(\mho,\vartheta)\geq 0$,
2. $\Omega(\mho,\vartheta)=0$ iff $\mho=$
3. $\Omega(\mho,\vartheta)=\Omega(\vartheta,\mho)$,
4. $\Omega(\mho,\vartheta)\leq\Omega(\mho,u)+\Omega(u,v)+\Omega(v,\vartheta)$

For all $\mho,\vartheta,u,v\in\aleph$.

Definition 2.2: [10] A sequence $\{\mho_n\}$ in $RMS(\aleph,\Omega)$ is said to converge if there is a point $\mho\in\aleph$ and for every $\in>0$ there exists $N\in\mathbb{N}$ such that $\Omega(\mho_n,\mho)<\in$ for every $n>N$.

Definition 2.3: [10] A sequence $\{\mho_n\}$ in a $RMS(\aleph,\Omega)$ is Cauchy if for every $\in>0$ there exists $N\in\mathbb{N}$ such that $\Omega(\mho_n,\mho_m)<\in$ for every $n,m>N$.

Definition 2.4: [10] $RMS(\aleph,\Omega)$ is said to be complete if every Cauchy sequence is convergent.

Definition 2.5: [27] Let $f:\aleph\to\aleph$ and $\alpha:\aleph\times\aleph\to[0,\infty)$. We say that f is an α-admissible mapping if $\alpha(\mho,\vartheta)\geq 1$ implies $\alpha(f\mho,f\vartheta)\geq 1,\mho,\vartheta\in\aleph$.

Main Results

Theorem 3.1: Let $(\aleph,\Omega)$ be a complete RMS and $T:\aleph\to\aleph$ be an α - admissible mapping. Assume that there exists a function $\beta:[0,\infty)\to[0,1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n)\to 1$ implies $t_n\to 0$ and

$$(\alpha(\mho,T\mho)\cdot\alpha(\vartheta,T\vartheta)+1)^{\Omega(T\mho,T\vartheta)}\leq 2^{\beta(M(\mho,\vartheta))M(\mho,\vartheta)}\ \forall\mho,\vartheta\in\aleph \text{ and } l\geq 1 \tag{3.1}$$

where: $M(\mho,\vartheta)=\max\{\Omega(\mho,\vartheta),\Omega(\mho,T\mho),\Omega(\vartheta,T\vartheta),\frac{\Omega(\mho,T\mho),\Omega(\vartheta,T\vartheta)}{\Omega(\mho,\vartheta)},\frac{\Omega(\mho,T\mho)(1+\Omega(\vartheta,T\vartheta))}{1+\Omega(\mho,\vartheta)}\}$

Suppose that if T is continuous and there exists $\mho_0\in\aleph$ such that $\alpha(\mho_0,T\mho_0)\geq 1$, then T has a fixed point.

Proof Let $\mho_0\in\aleph$ such that $\alpha(\mho_0,T\mho_0)\geq 1$. Construct a sequence $\{\mho_n\}$ in $\aleph$ as $\mho_{n+1}=T\mho_n$, $\forall n\in\mathbb{N}$.

If $\mho_{n+1}=\mho_n$, for some $n\in\mathbb{N}$, then $T\mho_n=\mho_n$ and we are done.

So, we suppose that $\Omega(\mho_n,\mho_{n+1})>0,\forall n\in\mathbb{N}$.

Since T is α-admissible, there exists $\mho_0\in\aleph$ such that $\alpha(\mho_0,T\mho_0)\geq 1$ which implies $\alpha(\mho_0,\mho_1)\geq 1$.

Similarly, we can say that $\alpha(\mho_1,\mho_2)=\alpha(T\mho_0,T^2\mho_0)\geq 1$.

By continuing this process, we get

$$\alpha(\mho_n,\mho_{n+1})\geq 1,\ \forall n\in\mathbb{N} \tag{3.2}$$

By using equation (3.2), we have

$$\begin{aligned} 2^{\Omega(T\mho_{n-1},T\mho_n)} &\leq\left(\alpha(\mho_{n-1},T\mho_{n-1})\alpha(\mho_n,T\mho_n)+1\right)^{\Omega(T\mho_{n-1},T\mho_n)}\\ &\leq 2^{\beta(M(\mho_{n-1},\mho_n))M(\mho_{n-1},\mho_n)}\end{aligned}$$

Now using equation (3.1), we get

$$\Omega(\mho_n,\mho_{n+1})\leq\beta(M(\mho_{n-1},\mho_n))M(\mho_{n-1},\mho_n), \tag{3.3}$$

Where

$$\begin{aligned} &M(\mho_{n-1},\mho_n)=\\ &\max\left\{\begin{matrix}\Omega(\mho_{n-1},\mho_n),\Omega(\mho_{n-1},T\mho_{n-1}),\Omega(\mho_n,T\mho_n),\frac{\Omega(\mho_{n-1},T\mho_{n-1})\Omega(T\mho_n,\mho_n)}{\Omega(\mho_{n-1},\mho_n)}\\ \frac{\Omega(\mho_{n-1},T\mho_{n-1})(1+\Omega(T\mho_n,\mho_n))}{1+\Omega(\mho_{n-1},\mho_n)}\end{matrix}\right\}\\ &=\max\{\Omega(\mho_{n-1},\mho_n),\Omega(\mho_{n-1},\mho_n),\Omega(\mho_n,\mho_{n+1})\}\end{aligned}$$

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Assume that if possible $\Omega\left(\mho_n,\mho_{n+1}\right) > \Omega\left(\mho_{n-1},\mho_n\right)$.

Then, $M\left(\mho_{n-1},\mho_n\right) = \Omega\left(\mho_n,\mho_{n+1}\right)$.

Using this in equation (3.3), we get

$$\Omega\left(\mho_n,\mho_{n+1}\right) < \beta\left(\Omega\left(\mho_n,\mho_{n+1}\right)\right)\Omega\left(\mho_n,\mho_{n+1}\right) \tag{3.4}$$

$\Rightarrow \Omega\left(\mho_n,\mho_{n+1}\right) < \Omega\left(\mho_n,\mho_{n+1}\right)$, which is a contradiction.

So $\Omega\left(\mho_n,\mho_{n+1}\right) \le \Omega\left(\mho_{n-1},\mho_n\right), \forall n$.

It follows that the sequence $\left\{\Omega\left(\mho_n,\mho_{n+1}\right)\right\}$ is a monotonically decreasing sequence of positive real numbers. So, it is convergent and suppose that $\lim_{n\to\infty}\Omega\left(\mho_n,\mho_{n+1}\right) = d$. Clearly, $d \ge 0$.

Claim: $d = 0$.

Equation (3.4) implies that

$$\frac{\Omega\left(\mho_n,\mho_{n+1}\right)}{\Omega\left(\mho_{n-1},\mho_n\right)} \le \beta\left(\Omega\left(\mho_{n-1},\mho_n\right) \le 1\right.$$

Which implies that $\lim_{n\to\infty}\beta\left(\Omega\left(\mho_{n-1},\mho_n\right) = 1\right.$.

Using the property of the function β, we conclude that $d = 0$, that is

$$\lim_{n\to\infty}\Omega\left(\mho_n,\mho_{n+1}\right) = 0. \tag{3.5}$$

In the similar way, we can prove that

$$\lim_{n\to\infty}\Omega\left(\mho_n,\mho_{n+2}\right) = 0 \tag{3.6}$$

Now, we will show that $\{\mho_n\}$ is a Cauchy sequence. Suppose, to the contrary that $\{\mho_n\}$ is not a Cauchy sequence. Then there exists $\in > 0$ and sequences $m(k)$ and $n(k)$ such that for all positive integers k, we have $n(k) > m(k) > k, \Omega\left(\mho_{n(k)},\mho_{m(k)}\right) \ge \epsilon$ and $\Omega\left(\mho_{n(k)},\mho_{m(k)-1}\right) < \epsilon$.

By the triangle inequality, we have

$$\begin{aligned} \in \le \Omega\left(\mho_{n(k)},\mho_{m(k)}\right) &\le \Omega\left(\mho_{n(k)},\mho_{m(k)-1}\right) + \Omega\left(\mho_{m(k)-1},\mho_{m(k)+1}\right) + \Omega\left(\mho_{m(k)-1},\mho_{m(k)}\right) \\ &< \in + \Omega\left(\mho_{m(k)-1},\mho_{m(k)+1}\right) + \Omega\left(\mho_{m(k)-1},\mho_{m(k)}\right), \end{aligned}$$

for all $k \in \mathbb{N}$.

Taking the limit as $k \to +\infty$ in the above inequality and using equations (3.5) and (3.6), we get

$$\lim_{k\to+\infty}\Omega\left(\mho_{n(k)},\mho_{m(k)}\right) = \epsilon. \tag{3.7}$$

Again, by triangle inequality, we have

$$\Omega\left(\mho_{n(k)},\mho_{m(k)}\right) - \Omega\left(\mho_{m(k)-1},\mho_{m(k)}\right) - \Omega\left(\mho_{n(k)-1},\mho_{n(k)}\right) \le \Omega\left(\mho_{n(k)-1},\mho_{m(k)-1}\right)$$

$$\Omega\left(\mho_{n(k)-1},\mho_{m(k)-1}\right) \le \Omega\left(\mho_{m(k)},\mho_{m(k)-1}\right) + \Omega\left(\mho_{n(k)},\mho_{m(k)}\right) + \Omega\left(\mho_{n(k)-1},\mho_{n(k)}\right).$$

Taking the limit as $k \to +\infty$, together with (3.5) - (3.7), we deduce that

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$$\lim_{k\to+\infty}\Omega\left(\mho_{n(k)-1},\mho_{m(k)-1}\right)=\epsilon. \tag{3.8}$$

From equations (3.1), (3.2), (3.6) and (3.8), we get

$$2^{\Omega\left(\mho_{n(k)+1},\mho_{m(k)+1}\right)}\le\left(\alpha\left(\mho_{n(k)},T\mho_{n(k)}\right)\alpha\left(\mho_{m(k)},T\mho_{m(k)}\right)+1\right)^{\Omega\left(\mho_{n(k)+1},\mho_{m(k)+1}\right)},$$

$$=\left(\alpha\left(\mho_{n(k)},T\mho_{n(k)}\right)\alpha\left(\mho_{m(k)},T\mho_{m(k)}\right)+1\right)^{\Omega\left(T\mho_{n(k)},T\mho_{m(k)}\right)}$$

$$\le 2^{\beta\left(M\left(\mho_{n(k)},\mho_{m(k)}\right)\right)M\left(\mho_{n(k)},\mho_{m(k)}\right)} \tag{3.9}$$

$$M\left(\mho_{n(k)-1},\mho_{m(k)-1}\right)= max\left\{\begin{array}{c}\Omega\left(\mho_{n(k)-1},\mho_{m(k)-1}\right),\Omega\left(\mho_{n(k)-1},\mho_{n(k)}\right),\Omega\left(\mho_{m(k)-1},\mho_{m(k)}\right),\\ \dfrac{\Omega\left(\mho_{n(k)-1},T\mho_{n(k)-1}\right)\Omega\left(T\mho_{m(k)-1},\mho_{m(k)-1}\right)}{\Omega\left(\mho_{n(k)-1},\mho_{m(k)-1}\right)},\dfrac{\Omega\left(\mho_{n(k)-1},T\mho_{n(k)-1}\right)\left(1+\Omega\left(T\mho_{m(k)-1},\mho_{m(k)-1}\right)\right)}{1+\Omega\left(\mho_{n(k)-1},\mho_{m(k)-1}\right)}\end{array}\right\},$$

$$= max\left\{\begin{array}{c}\Omega\left(\mho_{n(k)-1},\mho_{m(k)-1}\right),\Omega\left(\mho_{n(k)-1},\mho_{n(k)}\right),\Omega\left(\mho_{m(k)-1},\mho_{m(k)}\right),\dfrac{\Omega\left(\mho_{n(k)-1},\mho_{n(k)}\right)\cdot\Omega\left(\mho_{m(k)-1},\mho_{m(k)}\right)}{\Omega\left(\mho_{n(k)-1},\mho_{m(k)-1}\right)},\\ \dfrac{\Omega\left(\mho_{n(k)},\mho_{n(k)-1}\right)\left(1+\Omega\left(\mho_{m(k)-1},\mho_{m(k)}\right)\right)}{1+\Omega\left(\mho_{n(k)-1},\mho_{m(k)-1}\right)}\end{array}\right\}$$

Taking $k\to\infty$, we have

$$M\left(\mho_{n(k)-1},\mho_{m(k)-1}\right)=\max\{\epsilon,0,0,0,0\}$$

So, equation (9) implies that

$$\Omega\left(\mho_{n(k)+1},\mho_{m(k)+1}\right)\le\beta\left(M\left(\mho_{n(k)},\mho_{m(k)}\right)\right)M\left(\mho_{n(k)},\mho_{m(k)}\right)\le 1$$

Letting $k\to\infty$, we get

$$\lim k\to\infty\beta\left(\Omega\left(\mho_{n(k)},\mho_{m(k)}\right)\right)=1$$

By using definition of β function, we get

$$\Rightarrow \lim k\to\infty\Omega\left(\mho_{n(k)},\mho_{m(k)}\right)=0<\in,$$

which is a contradiction.

Hence, $\{\mho_n\}$ is a Cauchy sequence.

Since $(\aleph,\Omega)$ is a complete space, so $\{\mho_n\}$ is convergent and assume that $\mho_n\to\mho$ as $n\to\infty$.

Since T is continuous, then we have

$$T\mho=\lim_{n\to\infty}T\mho_n=\lim_{n\to\infty}\mho_{n+1}=\mho$$

So, $\mho$ is a fixed point of T.

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Theorem 3.2: Assume that all the hypothesis of Theorem 3.1 hold. Adding the following condition:

If $= T\mho$, then $\alpha(\mho, T\mho) \geq 1$

We obtain the uniqueness of fixed point.

Proof: Let z and z^* be two distinct fixed point of T in the setting of Theorem 3.1 and above defined condition holds, then $\alpha(z,Tz) \geq 1$ and $\alpha\left(z^*, Tz^*\right) \geq 1$

So,

$$2^{\Omega\left(Tz,Tz^*\right)} \leq \left(1+\alpha(z,Tz)\alpha\left(z^*,Tz^*\right)\right)^{\Omega\left(Tz,Tz^*\right)}$$

$$\leq 2^{\beta\left(M\left(z,z^*\right)\right)M\left(z,z^*\right)} \tag{3.10}$$

Where $M\left(z,z^*\right) = \max\left\{\begin{array}{l}\Omega\left(z,z^*\right), \Omega(Tz,z), \Omega\left(Tz^*,z\right), \\ \dfrac{\Omega(z,Tz)\cdot\Omega\left(Tz^*,z^*\right)}{\Omega\left(z,z^*\right)}, \dfrac{\Omega(z,Tz)\left(1+\Omega\left(Tz^*,z^*\right)\right)}{1+\Omega\left(z,z^*\right)}\end{array}\right\}$

$= \Omega\left(z,z^*\right)$.

So, equation (3.10) implies

$$\Omega\left(z,z^*\right) = \Omega\left(Tz,Tz^*\right) \leq \beta\left(\Omega\left(z,z^*\right)\right)\Omega\left(z,z^*\right)$$

$$\Rightarrow \beta\left(\Omega\left(z,z^*\right)\right) = 1$$

$$\Rightarrow \Omega\left(z,z^*\right) = 0 \Rightarrow z = z^*$$

Corollary 3.3: Let $(\aleph, \Omega)$ be a complete RMS and $T:\aleph \to \aleph$ be an α-admissible mapping. Assume that there exists a function $\beta:[0,\infty) \to [0,1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ and $(\alpha(\mho,T\mho)\cdot\alpha(\vartheta,T\vartheta)+1)^{\Omega(T\mho,T\vartheta)} \leq 2^{\beta(\Omega(\mho,\vartheta))\Omega(\mho,\vartheta)}$ for all $\mho, \vartheta \in \aleph$ where $l \geq 1$. Suppose that if T is continuous and there exists $\mho_0 \in \aleph$ such that $\alpha\left(\mho_0, T\mho_0\right) \geq 1$, then T has a fixed point.

Proof: Taking $M(\mho,\vartheta) = \Omega(\mho,\vartheta)$ in Theorem 3.1, one can get the proof.

Corollary 3.4. Assume that all the hypotheses of Corollary 3.3 hold. Adding the following condition:

(a) If $\mho = T\mho$, then $\alpha(\mho, T\mho) \geq 1$,

we obtain the uniqueness of the fixed point of T.

Proof: Taking $M(\mho,\vartheta) = \Omega(\mho,\vartheta)$ in Corollary 3.3.

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