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Research Article

Coincidence and common fixed points for F-Contractive mappings

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Abstract

The purpose of this article is to establish the existence and uniqueness of coincidence and common fixed point of discontinuous non-compatible faintly compatible pair of self maps in non-complete metric space without using containment requirement of range space of involved maps satisfying Ciric type *F*-contraction and Hardy-Roger type *F*-contraction. Some illustrative examples associated with pictographic validations are provided to demonstrate the main results and to show the genuineness of our results. We consider the application of our results to the study of a two-point boundary value problem related to second order differential equation, solve the two-point boundary value problem of the second-order differential equation arising in electric circuit equation, and also apply our results to Volterra type integral equation using Ciric type *F*-contraction as well as Hardy Roger type *F*-contraction.

Mathematics subject classification: 47H10; 54H25; 54E50

Introduction

The knowledge and applications of fixed point theory are fundamental to the growth of Functional analysis, Mathematics, and Sciences in general. Banach in 1922, formulated a pivotal result of metric fixed point theory that is simple and well defined which is now popularly referred to as Banach contraction principle. This principle is a very popular tool for guaranteeing the existence and uniqueness of solutions of multitude problems arising in several domains of Mathematics and Physical Sciences. Several extensions of this core result are available in the existing literature on metric fixed point theory. In the course of the last several decades, this principle has been extended and generalized in many areas with several applications in various directions, some of which can be seen in [1-8].

Wardowski [9] introduced the concept of F-contractive mapping on a metric space and proved a fixed point theorem for such a map on a complete metric space.

Bisht and Shahzad [10] introduced the concept of faint compatibility, which they viewed as an improvement on the notion of conditional compatibility introduced by Pant and Bisht [11]. This new concept allowed for the existence of a broader set of points, including common fixed points, multiple common fixed points, coincidence points, and multiple coincidence points.

Tomar, et al. [12] extended the notion of faint compatibility to a hybrid pair of maps. However, it is well known that weak compatibility is the most widely used concept among all weaker forms of commuting maps in fixed-point considerations but is not applicable when a pair of self-maps has more than one coincidence point. Tomar A. and Sharma R. [13] used the notion of F-contraction introduced by Wardowski to establish coincidence and common fixed point theorems for a discontinuous noncompatible pair of self-maps in noncomplete metric space.

In view of the above considerations, we present some results by reviewing the works in the literature by establishing some coincidence and common fixed point theorems for discontinuous non-compatible faintly compatible pair of self maps in non-complete metric space without using containment requirement of range space of involved maps satisfying Ciri´c type F-contraction and Hardy-Roger type F-contraction. Some illustrative examples associated with pictographic validations to demonstrate the main results and to show the genuineness of our results are provided. We consider the application of our results to the study of a two-point boundary value problem related to second order differential equation, solve the two-point boundary value problem of the second-order differential equation arising in electric circuit equation, and also apply our results to Volterra type integral equation using Ciri´c type F-contraction as well as Hardy Roger type F-contraction. The results presented in this paper extend and generalize several existing results in the current literature on the topic. The new findings build upon and extend the knowledge of the field, providing additional insight and applications for future research.

Throughout the article, we denoted that R is the set of all real numbers, R^+ is the set of all positive real numbers and $\mathbb{N}$ is the set of natural numbers.

Preliminaries

We start this section by presenting the notion of F-contraction with some relevant definitions and examples.

Definition 2.1 [14] Let (X, d) be a metric space. A map $f : X \to X$ is an F-contraction if there exists $\tau > 0$ such that

$$\tau + F\left(d\left(fx, fy\right)\right) \le F\left(d\left(x, y\right)\right) \quad (2.1)$$

for all $x, y \in X$ with $fx \neq fy$, where F: $R^+ \to R$ is a function satisfying:

(i) F is strictly increasing, i.e., for all $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$;

(ii) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F\left(\alpha_n\right) = -\infty$;

(iii) There exists $k \in (0,1)$ such that $\lim_{n\to 0^+} \alpha^k F\left(\alpha\right) = 0$.

We denote F, the family of all functions F: $R^+ \to R$ satisfying the conditions (i)-(iii). Every F-contraction is a contractive map i.e.,

$$d\left(fx, fy\right) < d\left(x, y\right)$$

for all $x, y \in X$, $fx \neq fy$ and hence is necessarily continuous.

Note that Banach contraction is a particular case of F-contraction. Meanwhile there exist F-contractions, which are not Banach contractions (Wardowski [9]). Taking different functions F, we obtain a variety of F-contractions, some of them being already known in the literature.

Some examples of the functions belonging to F are:

(i) $F\left(\alpha\right) = \ln \alpha$;

(ii) $F\left(\alpha\right) = \ln \alpha + \alpha, \alpha > 0$;

(iii) $F\left(\alpha\right) = \frac{-1}{\sqrt{\alpha}}, \alpha > 0$;

(iv) $F\left(\alpha\right) = \ln\left(\alpha^2 + \alpha\right), \alpha > 0$.

Definition 2.2 [4] Let (X, d) be a metric space and $f, g : X \to X$. A pair (f,g) on a metric space (X, d) is Compatible, if $\lim_{n\to\infty} d\left(fgx_n, gfx_n\right) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u, \quad (2.2)$$

for some $u \in X$.

Definition 2.3 [15] Let (X, d) be a metric space and $f, g : X \to X$. A pair (f,g) on a metric space (X, d) is non-compatible, if (f,g)

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is not compatible, i.e., if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$, but either $\lim_{n\to\infty} d\left(fgx_n, gfx_n\right) \neq 0$ or non-existent.

Definition 2.4 [16] Let (X, d) be a metric space and $f, g : X \to X$. A pair (f,g) on a metric space (X, d) is weakly compatible, if the pair commute on the set of their coincidence points, i.e., for $x \in X$, $fx = gx$ implies $fgx = gfx$.

Definition 2.5 [11] Let (X, d) be a metric space and $f, g : X \to X$. A pair (f,g) on a metric space (X, d) is conditionally compatible, if whenever the set of sequences $\{x_n\}$ satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ is non-empty, there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = u$, for some $u \in X$ and $\lim_{n\to\infty} d\left(fgy_n, gfy_n\right) = 0$.

Definition 2.6 [17] Let (X, d) be a metric space and $f, g : X \to X$. A pair (f,g) on a metric space (X, d) is reciprocally continuous, if $\lim_{n\to\infty} fgx_n = fx$, $\lim_{n\to\infty} gfx_n = gx$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$, for some $u \in X$.

Definition 2.7 [18] Let (X, d) be a metric space and $f, g : X \to X$. A pair (f,g) on a metric space (X, d) is conditionally reciprocally continuous, if whenever the set of sequences $\{x_n\}$ satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ is non-empty, there exists a sequence $\{y_n\}$ satisfying $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = u$ (say) for some $u \in X$ such that $\lim_{n\to\infty} fgy_n = fu$ and $\lim_{n\to\infty} gfy_n = gu$.

Definition 2.8 [10] Let (X, d) be a metric space and $f, g : X \to X$. A pair (f, g) on a metric space (X, d) is faintly compatible, if (f, g) is conditional compatible and f and g commute on a non-empty subset of the set of coincidence points, whenever the set of coincidence points is nonempty.

Definition 2.9 [19] Let (X, d) be a metric space and $f, g : X \to X$. A pair of self maps f and g have a coincidence point at x g X if $fx = gx$. Further, a point $x \in X$ is a common fixed point of f and g if $fx = gx = x$.

Main results

Definition 3.1 Let $f, g : X \to X$ be a pair of self-maps of a metric space (X, d) is said to be Ciri´c type generalized F-contraction if there exist $F \in F$ and $\tau > 0$ such that for all $x, y \in X$:

$$d\left(fx, fy\right) > 0 \Rightarrow \tau + F\left(d\left(fx, fy\right)\right) \leq F\left(M\left(x, y\right)\right),$$

Where

$$M\left(gx, gy\right) = max\left\{\begin{array}{l} d\left(gx, gy\right), d\left(gx, fy\right), d\left(gy, fx\right), d\left(gx, fx\right), d\left(gy, fy\right), \dfrac{d\left(gx, fx\right)d\left(gy, fy\right)}{d\left(gx, gy\right)}, \\ \dfrac{d\left(gx, fx\right)d\left(gy, fy\right)}{d\left(gx, gy\right) + d\left(gx, fy\right) + d\left(gy, fx\right)}, \dfrac{d\left(gx, fx\right)d\left(gx, fy\right) + d\left(gy, fx\right)d\left(gy, fy\right)}{d\left(gy, fx\right) + d\left(gx, fy\right)} \end{array}\right\} \quad (3.1)$$

Remark 3.2 Every Ciri´c type generalized F-contraction for a pair of self maps is also a F-contraction but the reverse implication does not always hold.

Ciri´c type generalized F-contraction for a faintly compatible pair of maps using conditional reciprocal continuity which is weaker than the continuity of even a single map.

Theorem 3.3 Let $f, g : X \to X$ be a faintly compatible pair of self-maps of a metric space (X, d) be conditionally reciprocally continuous. Then f and g have a coincidence point. Moreover, f and g have a unique common fixed point provided that the pair of self-maps (f,g) satisfies Ciri´c type generalized F-contraction.

Proof Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t,$$

for some $t \in X$. Since pair (f,g) is faintly compatible, there exists a sequence $\{y_n\}$ in X satisfying

$$\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = u,$$

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for some $u \in X$. such that

$$\lim_{n\to\infty}\left(fgy_n, gfy_n\right) = 0.$$

Since the pair (f, g) is also conditionally reciprocally continuous, we get

$$\lim_{n\to\infty} fgy_n = fu$$

and

$$\lim_{n\to\infty} gfy_n = gu.$$

Hence, $fu = gu$, i.e., f and g have a coincidence point.

Also, since the pair (f, g) is faintly compatible, we get

$fgu = gfu.$

So $ffu = g\,fu = ggu$. If $fu \neq ffu$, then by (3.1) with $x = u$ and $y = fu$, we have

$$\tau + F\left(d\left(fu, ffu\right)\right) \le F\left(\max\left\{\begin{array}{l} d\left(gu, gfu\right), d\left(gu, ffu\right), d\left(gfu, fu\right), d\left(gu, fu\right), d\left(gfu, ffu\right), \dfrac{d\left(gu, fu\right)d\left(gfu, ffu\right)}{d\left(gu, gfu\right)}, \\ \dfrac{d\left(gu, fu\right), d\left(gfu, ffu\right)}{d\left(gu, gfu\right)+d\left(gu, ffu\right)+d\left(gfu, fu\right)}, \dfrac{d\left(gu, fu\right)d\left(gu, ffu\right)+d\left(gfu, fu\right)d\left(gfu, ffu\right)}{d\left(gfu, fu\right)+d\left(gu, ffu\right)} \end{array}\right\}\right) = F\left(\max\left\{d\left(fu, ffu\right), d\left(fu, ffu\right), d\left(ffu, fu\right)\right\}\right) \tag{3.2}$$

i.e.,

$$\tau + F\left(d\left(fu, ffu\right)\right) \le F\left(d\left(fu, ffu\right)\right)$$

a contradiction. Hence, fu is a common fixed point of f and g.

Uniqueness: Assume that fu^* is another common fixed point of f and g. Then, we have

$fu^* = ffu = ffu = fu.$

Hence, $fu^* = fu$.

Example 3.4 Let $X = (4,12)$ and d be the usual metric on X. Define $f, g : X \to X$ as follows:

$$fx = \begin{cases} 5, & x \le 5 \\ 7, & x > 5, \end{cases} \quad gx = \begin{cases} 10 - x, & x \le 5 \\ 9, & x > 5. \end{cases}$$

We check whether the pair (f, g) for compatible or non-compatible, and reciprocally continuous or not using definitions 2.2, 2.3, and 2.6.

Let $\{x_n\}$ be a sequence in X such that $x_n = 5 - \frac{1}{n}$, then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 5$ and $\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} f\left(5 + \frac{1}{n}\right) = 7$, $\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} g5 = 5$, i.e., $\lim_{n\to\infty} d\left(fgx_n, gfx_n\right) \neq 0$, i.e., the pair (f, g) is non-compatible. Also $\lim_{n\to\infty} fgx_n = 7 \neq f5$, i.e., pair (f, g) is not reciprocally continuous.

Also, check that the pair (f, g) is conditionally compatible and faintly compatible using definitions 2.5 and 2.8.

Let $\{y_n\}$ be a sequence in X such that $y_n = 5$, then $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = 5$ and $\lim_{n\to\infty} gfy_n = 5$, $\lim_{n\to\infty} fgy_n = 5$, i.e., $\lim_{n\to\infty} d\left(gfy_n, fgy_n\right) = 0$. Therefore, pair (f, g) is conditionally compatible. Also $f5 = g5$, $fg5 = gf5$, i.e., pair (f, g) is faintly compatible.

Further, check that the pair (f,g) is conditionally reciprocally continuous using definition 2.7. $\lim_{n\to\infty} fgy_n = 5 = f5$, $\lim_{n\to\infty} gfy_n = 5 = g5$, pair (f, g) is conditionally reciprocally continuous.

In addition, f and g satisfy Ciri´c type F-contraction for $\tau = 0.002$ and $F(\alpha) = \log \alpha$.

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Hence, all the conditions of Theorem 3.3 are satisfied and $x = 5$ is a unique coincidence and common fixed point of f and g. Also, both the self-maps are discontinuous at a common fixed point and are neither compatible nor reciprocally continuous. Further $fX \not\subseteq gX$. (Figure 1).

We have the following corollaries for Ciri´c type generalized F-contraction:

Corollary 3.5 Let $f, g : X \to X$ be a faintly compatible pair of self-maps of a metric space (X, d) be conditionally reciprocally continuous. Then f and g have a coincidence point. Moreover, f and g have a unique common fixed point provided that the pair of self-maps (f, g) satisfies:

$$d\left(fx, fy\right) > 0 \Rightarrow \tau + F\left(d\left(fx, fy\right)\right) \le F\left(M\left(x, y\right)\right),$$

Where

$$M\left(gx, gy\right) = max\left\{\begin{array}{c} d\left(gx, gy\right), d\left(gx, fy\right), d\left(gy, fx\right), d\left(gx, fx\right), \\ d\left(gy, fy\right), \dfrac{d\left(gx, fx\right)d\left(gy, fy\right)}{d\left(gx, gy\right)}, \dfrac{d\left(gx, fx\right)d\left(gy, fy\right)}{d\left(gx, gy\right) + d\left(gx, fy\right) + d\left(gy, fx\right)} \end{array}\right\} \quad (3.3)$$

Corollary 3.6 Let $f, g : X \to X$ be a faintly compatible pair of self-maps of a metric space (X, d) be conditionally reciprocally continuous. Then f and g have a coincidence point. Moreover, f and g have a unique common fixed point provided that the pair of self-maps (f, g) satisfies:

$$d\left(fx, fy\right) > 0 \Rightarrow \tau + F\left(d\left(fx, fy\right)\right) \le F\left(M\left(x, y\right)\right),$$

where

$$M\left(gx, gy\right) = max\left\{\begin{array}{c} d\left(gx, gy\right), d\left(gx, fy\right), d\left(gy, fx\right), d\left(gx, fx\right), d\left(gy, fy\right), \\ \dfrac{d\left(gx, fx\right)d\left(gy, fy\right)}{d\left(gx, gy\right)}, \dfrac{d\left(gx, fx\right)d\left(gx, fy\right) + d\left(gy, fx\right)d\left(gy, fy\right)}{d\left(gy, fx\right) + d\left(gx, fy\right)} \end{array}\right\} \quad (3.4)$$

Corollary 3.7 Let $f, g : X \to X$ be a faintly compatible pair of self-maps of a metric space (X, d) be conditionally reciprocally continuous. Then f and g have a coincidence point. Moreover, f and g have a unique common fixed point provided that the pair of self-maps (f, g) satisfies:

$$d\left(fx, fy\right) > 0 \Rightarrow \tau + F\left(d\left(fx, fy\right)\right) \le F\left(M\left(x, y\right)\right),$$

where

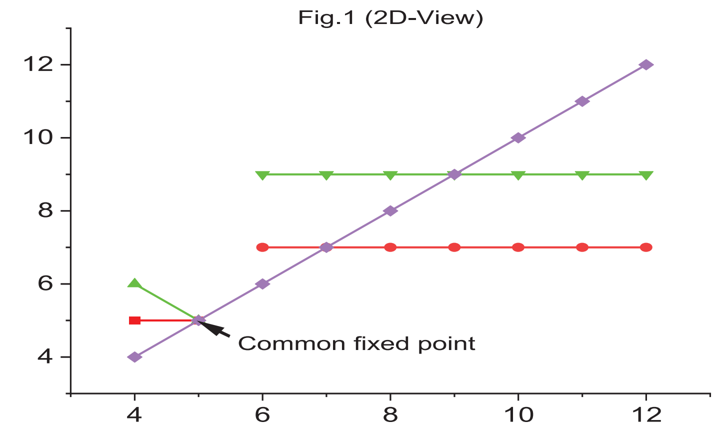

Figure 1: 2D-view. The red line denotes fx, the green line denotes gx and the purple line denotes the line $x = y$. Clearly, the functions f and g intersect on the line $x = y$ only at $x = 5$, i.e., $x = 5$ is the unique common fixed point of f and g.

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$$M(gx,gy)=max\left\{\begin{matrix} d(gx,gy),d(gx,fy),d(gy,fx), \\ d(gx,fx),d(gy,fy),\dfrac{d(gx,fx)d(gy,fy)}{d(gx,gy)} \end{matrix}\right\} \quad (3.5)$$

Now, we extend Hardy-Rogers-type F-contraction introduced by Cosentino and Vetro [20] to a pair of maps.

Definition 3.8 Let $f, g: X \to X$ be a faintly compatible pair of self-maps of a metric space (X, d) The pair of self-maps $(f, \in)$ of on a metric space (X, d) is said to be Hardy-Rogers-type generalized F-contraction if there exist $F \in F$ and $\tau > 0$ such that for all $x, y \in X$:

$$d(fx,fy)>0 \Rightarrow \tau+F(d(fx,fy)) \le F(M(x,y)),$$

Where

$$M(gx,gy)=\alpha d(gx,gy)+\beta\left[d(gx,fy)+d(gy,fx)\right]+\gamma\left[d(gx,fx)+d(gy,fy)\right]+\delta\frac{d(gx,fx)d(gy,fy)}{d(gx,gy)}$$
$$+\lambda\frac{d(gx,fx)d(gy,fy)}{d(gx,gy)+d(gx,fy)+d(gy,fx)}+\mu\frac{d(gx,fx)d(gx,fy)+d(gy,fx)d(gy,fy)}{d(gy,fx)+d(gx,fy)} \quad (3.6)$$

for $\alpha,\beta,\gamma,\delta,\lambda,\mu \ge 0, \alpha+2(\beta+\gamma)+\delta+\lambda+\mu<1..$

Remark 3.9 Every Hardy-Rogers-type F-contraction for a pair of self maps is also a F-contraction but the reverse implication does not always hold.

Now, we prove the next result using Hardy-Rogers-type F-contraction.

Theorem 3.10 Let $f, g: X \to X$ be a faintly compatible pair of self-maps on a metric space (X, d) be conditionally reciprocally continuous. Then f and g have a coincidence point. Moreover, f and g have a unique common fixed point provided that the pair of self-maps (f, g) satisfies Hardy-Rogers-type generalized F-contraction.

Proof For all $x, y \in X$, we have

$$\alpha d(gx,gy)+\beta\left[d(gx,fy)+d(gy,fx)\right]+\gamma\left[d(gx,fx)+d(gy,fy)\right]+\delta\frac{d(gx,fx)d(gy,fy)}{d(gx,gy)}+$$
$$\lambda\frac{d(gx,fx)d(gy,fy)}{d(gx,gy)+d(gx,fy)+d(gy,fx)}+\mu\frac{d(gx,fx)d(gx,fy)+d(gy,fx)d(gy,fy)}{d(gy,fx)+d(gx,fy)}$$
$$\le(\alpha+2(\beta+\gamma)+\delta+\lambda+\mu)max\left\{\begin{matrix} d(gx,gy),d(gx,fy),d(gy,fx),d(gx,fx),d(gy,fy), \\ \dfrac{d(gx,fx)d(gy,fy)}{d(gx,gy)},\dfrac{d(gx,fx)d(gy,fy)}{d(gx,gy)+d(gx,fy)+d(gy,fx)}, \\ \dfrac{d(gx,fx)d(gx,fy)+d(gy,fx)d(gy,fy)}{d(gy,fx)+d(gx,fy)} \end{matrix}\right\}$$

$$<max\left\{\begin{matrix} d(gx,gy),d(gx,fy),d(gy,fx),d(gx,fx),d(gy,fy),\dfrac{d(gx,fx)d(gy,fy)}{d(gx,gy)}, \\ \dfrac{d(gx,fx)d(gy,fy)}{d(gx,gy)+d(gx,fy)+d(gy,fx)},\dfrac{d(gx,fx)d(gx,fy)+d(gy,fx)d(gy,fy)}{d(gy,fx)+d(gx,fy)} \end{matrix}\right\} \quad (3.7)$$

Now, we can have a similar proof to that of Theorem 3.3, by

Letting $\{x_n\}$ be a sequence in X such that

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$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t,$$

for some $t \in X$. Since pair (f, g) is also faintly compatible, there exists a sequence $\{y_n\}$ in X satisfying

$$\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = u,$$

for some $u \in X$ such that

$$\lim_{n\to\infty} \left(fgy_n, gfy_n\right) = 0.$$

As pair (f, g) is also conditionally reciprocally continuous, we get

$$\lim_{n\to\infty} fgy_n = fu$$

and

$$\lim_{n\to\infty} gfy_n = gu.$$

Hence $fu = gu$, i.e., f and g have a coincidence point. Since the pair (f, g) is faintly compatible, we get

$fgu = gfu$

So $ffu = fgu = gfu = ggu$. If $fu = ffu$, then by (3.7) with $x = u$ and $y = fu$, we have

$$\tau + F\left(d\left(fu, ffu\right)\right) \le F\left(max\left\{\begin{array}{l} d\left(gu, gfu\right), d\left(gu, ffu\right), d\left(gfu, fu\right), d\left(gu, fu\right), d\left(gfu, ffu\right), \dfrac{d\left(gu, fu\right)d\left(gfu, ffu\right)}{d\left(gu, gfu\right)}, \\ \dfrac{d\left(gu, fu\right), d\left(gfu, ffu\right)}{d\left(gu, gfu\right) + d\left(gu, ffu\right) + d\left(gfu, fu\right)}, \dfrac{d\left(gu, fu\right)d\left(gu, ffu\right) + d\left(gfu, fu\right)d\left(gfu, ffu\right)}{d\left(gfu, fu\right) + d\left(gu, ffu\right)} \end{array}\right\}\right) \quad (3.8)$$

$$= F\left(max\left\{d\left(fu, gfu\right), d\left(fu, ffu\right), d\left(ffu, fu\right)\right\}\right)$$

i.e.,

$$\tau + F\left(d\left(fu, ffu\right)\right) \le F\left(d\left(fu, ffu\right)\right)$$

a contradiction. Hence fu is a common fixed point of f and g.

Uniqueness: Assume also that fu^* is another common fixed point of f and g. Then, we have

$fu^* = ffu^* = ffu = fu.$

Hence, $fu^* = fu$.

Remark 3.11 It is interesting to point out here that Theorem 3.3 is an easy consequence of Theorem 3.10.

Example 3.12 Let $X = (3,8)$ and d be the usual metric in X. Define $f, g : X \to X$ as follows:

$$fx = \begin{cases} 4, & x \le 4 \\ 5, & x > 4, \end{cases} \quad gx = \begin{cases} 8 - x, & x \le 4 \\ 7, & x > 4. \end{cases}$$

We check whether the pair (f, g). is compatible or non-compatible and reciprocally continuous or not using definitions 2.2, 2.3, and 2.6.

Let $\{x_n\}$ be a sequence in X such that $x_n = 4 - \frac{1}{n}$, then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 4$ and $\lim_{n\to\infty} fgx_n = 5$, $\lim_{n\to\infty} gfx_n = 4$, i.e., $\lim_{n\to\infty} d\left(fgx_n, gfx_n\right) \ne 0$, i.e., pair (f, g) is non-compatible. Also $\lim_{n\to\infty} fgx_n = 5 \ne f4$, i.e., pair (f, g) is not reciprocally continuous.

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Also, check that the pair (f, g) is conditionally compatible and faintly compatible using definitions 2.5 and 2.8.

Let $\{y_n\}$ be a sequence in X such that $y_n = 4$, then $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = 4$ and $\lim_{n\to\infty} gfy_n = \lim_{n\to\infty} fgy_n = 4$, i.e., $\lim_{n\to\infty} d\left(gfy_n, fgy_n\right) = 0$

Therefore, pair (f, g) is conditionally compatible. Also $f4 = g4,\ fg4 = gf4$, i.e., pair (f, g) is faintly compatible.

Further, check that the pair (f, g) is conditionally reciprocally continuous using definition 2.7. $\lim_{n\to\infty} fgy_n = 4 = f4,\ \lim_{n\to\infty} gfy_n = 4 = g4$ pair (f, g) is conditionally reciprocally continuous.

In addition, f and g satisfy Hardy Roger type F-contraction (3.6) for $g = 0.02$ and $F(x) = \log x$, $\alpha = \frac{1}{3}, \beta = \frac{1}{8}, \gamma = \frac{1}{8}, \delta = \frac{1}{5}, \lambda = \frac{1}{5}, \mu = \frac{1}{5}$.

Hence, all the conditions of Theorem 3.10 are satisfied and $x = 4$ is a unique coincidence and common fixed point of f and e conditions of Theorem 3.10 are satisfied and $x = 4$ is a unique coinciden. Moreover, both the self maps are discontinuous at a common fixed point and are neither compatible nor reciprocally continuous. Further $fX \not\subseteq gX$. (Figure 2).

Also, we extend Hardy-Rogers-type F-contraction introduced by Cosentino and Vetro [20] to a pair of self-maps.

Definition 3.13 Let f, d are neither compatible nor reciprocally continuous. Fur : $X \to X$ be a faintly compatible pair of self-maps of a metric space (X, d) The pair of self-maps (f, d are neither compatible nor reciprocally continuous. Fur) is said to be weak Hardy-Rogers-type generalized F-contraction if there exist $F \in F$ and $\tau > 0$ such that for all $x, y \in X$:

$$d\left(fx, fy\right) > 0 \Rightarrow \tau + F\left(d\left(fx, fy\right)\right) \le F\left(M\left(x, y\right)\right),$$

Where

$$M\left(gx, gy\right) = \alpha d\left(gx, gy\right) + \beta\left[d\left(gx, fy\right) + d\left(gy, fx\right)\right] + \gamma\left[d\left(gx, fx\right) + d\left(gy, fy\right)\right] + \delta\frac{d\left(gx, fx\right)d\left(gy, fy\right)}{d\left(gx, gy\right)} \tag{3.9}$$

$$+\lambda\frac{d\left(gx, fx\right)d\left(gy, fy\right)}{d\left(gx, gy\right) + d\left(gx, fy\right) + d\left(gy, fx\right)} + \mu\frac{d\left(gx, fx\right)d\left(gx, fy\right) + d\left(gy, fx\right)d\left(gy, fy\right)}{d\left(gy, fx\right) + d\left(gx, fy\right)} + Ld\left(gx, fy\right)$$

For $\alpha, \beta, \gamma, , \lambda, \mu, L \ge 0$ such that $\alpha + 2(\beta +) + \delta + \lambda + \mu + L < 1$.

Theorem 3.14 Let $f, g : X \to X$ be a faintly compatible pair of self-maps of a metric space (X, d) be conditionally reciprocally continuous. Then f and g have a coincidence point. Moreover, f and g have a unique common fixed point provided that the pair of self-maps (f, g) satisfies weak Hardy-Rogers-type generalized F-contraction (3.9).

Proof The proof follows the same lines as in the case of Theorem 3.10.

Example 3.15 Let $X = (3,15)$ and d be the usual metric in X. Define $f, g : X \to X$ as follows:

$$fx = \begin{cases} 4, & x \le 4 \\ 8, & x > 4, \end{cases} \quad gx = \begin{cases} 8 - x, & x \le 4 \\ 14, & x > 4. \end{cases}$$

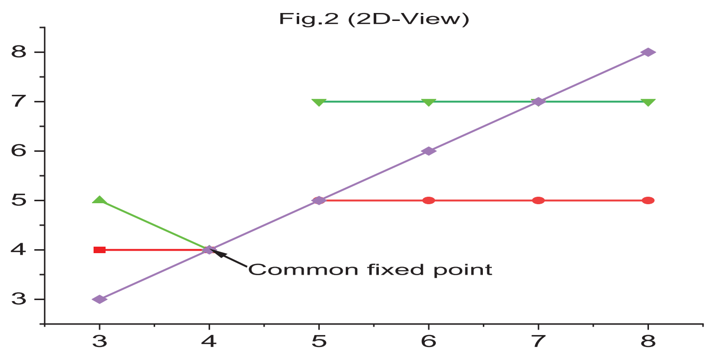

Figure 2: 2D-View. The red line denotes *fx* the green line denotes x and the purple line denotes the line *y = x*. Clearly, the functions *f* and intersect on the line *y = x* only at *x = 4*, i.e., *x = 4* is the unique common fixed point of *f* and *g*.

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We check whether the pair (f, g) is compatible or non-compatible and reciprocally continuous or not using definitions 2.2, 2.3, and 2.6.

Let $\{x_n\}$ be a sequence in X such that $x_n = 4 - \frac{1}{n}$, then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 4$ and, $\lim_{n\to\infty} fgx_n = 8$, $\lim_{n\to\infty} gfx_n = 4$, i.e., $\lim_{n\to\infty} d\left(fgx_n, gfx_n\right) \neq 0$, i.e., pair (f, g) is non-compatible. Also $\lim_{n\to\infty} fgx_n = 8 \neq f4$, i.e., pair (f, g) is not reciprocally continuous.

Also, check that the pair (f, g) is conditionally compatible and faintly compatible using definitions 2.5 and 2.8.

Let $\{y_n\}$ be a sequence in X such that $y_n = 4$, then $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = 4$ and $\lim_{n\to\infty} gfy_n = \lim_{n\to\infty} fgy_n = 4$, i.e., $\lim_{n\to\infty} d\left(gfy_n, fgy_n\right) = 0$. Therefore, pair (f, g) is conditionally compatible. Also $f4 = g4 = fg4 = gf4$, i.e., pair (f, g) is faintly compatible.

Further, check that the pair (f, g) is conditionally reciprocally continuous using definition 2.7. $\lim_{n\to\infty} fgy_n = 4 = f4$, $\lim_{n\to\infty} gfy_n = 4 = g4$, pair (f, g) is conditionally reciprocally continuous.

In addition, f and g satisfy Hardy-Rogers type F-contraction condition (3.9) for $\tau = \frac{1}{20}$ and $F(x) = -\frac{1}{\sqrt{x}}, \alpha = \frac{1}{9}, \beta = \frac{1}{3}, \gamma = \frac{1}{3}, \delta = \frac{1}{9}, \lambda = \frac{1}{9}, \mu = \frac{1}{9}, L = \frac{1}{9}$. Hence all the conditions of Theorem 3.14 are satisfied and $x = 4$ is a unique coincidence and common fixed point of f and g. Moreover, both the self-maps are discontinuous at a common fixed point and are neither compatible nor reciprocally continuous. Further $fX \not\subseteq gX$. (Figure 3).

We have the following corollaries for Hardy-Rogers-type generalized F-contraction:

Corollary 3.16 Let $f, g : X \to X$ be a faintly compatible pair of self-maps of a metric space (X, d) be conditionally reciprocally continuous. Then f and g have a coincidence point. Moreover, f and g have a unique common fixed point provided that the pair of self-maps (f, g) satisfies:

$$d(fx, fy) > 0 \Rightarrow \tau + F\left(d(fx, fy)\right) \leq F\left(M(x, y)\right),$$

Where

$$M(gx, gy) = \alpha d(gx, gy) + \beta\left[d(gx, fy) + d(gy, fx)\right] + \gamma\left[d(gx, fx) + d(gy, fy)\right] + \delta \frac{d(gx, fx)\, d(gy, fy)}{d(gx, gy) + d(gx, fy) + d(gy, fx)} \quad (3.10)$$

for $\alpha, \beta, \gamma, \delta \geq 0$ such that $\alpha + 2\beta + 2\gamma + \delta < 1$.

Corollary 3.17 Let $f, g : X \to X$ be a faintly compatible pair of self-maps of a metric space (X, d) be conditionally reciprocally continuous. Then f and g have a coincidence point. Moreover, f and g have a unique common fixed point provided that the pair of self-maps (f, g) satisfies:

$$d(fx, fy) > 0 \Rightarrow \tau + F\left(d(fx, fy)\right) \leq F\left(M(x, y)\right),$$

Where

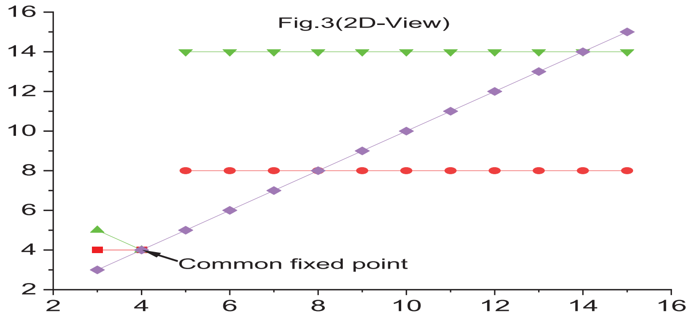

Figure 3: 2D- View. The red line denotes fx, the green line denotes gx and the purple line denotes the line $y = x$. Clearly, the functions f and g intersect on the line $y = x$ only at $x = 4$, i.e., $x = 4$ is the unique common fixed point of f and g.

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$$M(gx, gy) = \alpha d(gx, gy) + \beta\left[d(gx, fy) + d(gy, fx)\right] + \gamma\left[d(gx, fx) + d(gy, fy)\right] + \delta\frac{d(gx, fx)\,d(gy, fy)}{d(gx, gy)} \quad (3.11)$$

for $\alpha, \beta, \gamma, \delta \geq 0$ such that $\alpha + 2\beta + 2\gamma + \delta < 1$.

Corollary 3.18 Let f, g : X → X be a faintly compatible pair of self-maps of a metric space (X, d) be conditionally reciprocally continuous. Then f and g have a coincidence point. Moreover, f and g have a unique common fixed point provided that the pair of self-maps (f, g) satisfies:

$$d(fx, fy) > 0 \Rightarrow \tau + F(d(fx, fy)) \leq F(M(x, y)),$$

where

$$M(gx, gy) = \alpha d(gx, gy) + \beta\left[d(gx, fy) + d(gy, fx)\right] + \delta\frac{d(gx, fx)\,d(gy, fy)}{d(gx, gy)} + \lambda\frac{d(gx, fx)\,d(gy, fy)}{d(gx, gy) + d(gx, fy) + d(gy, fx)} \quad (3.12)$$

For $\alpha, \beta, \delta, \lambda \geq 0$ such that $\alpha + 2\beta + \delta + \lambda < 1$.

Corollary 3.19 Let f, g : X → X be a faintly compatible pair of self-maps of a metric space (X, d) be conditionally reciprocally continuous. Then f and g have a coincidence point. Moreover, f and g have a unique common fixed point provided that the pair of self-maps (f, g) satisfies:

$$d(fx, fy) > 0 \Rightarrow \tau + F(d(fx, fy)) \leq F(M(x, y)),$$

Where

$$M(gx, gy) = \alpha d(gx, gy) + \beta\left[d(gx, fy) + d(gy, fx)\right] + \gamma\left[d(gx, fx) + d(gy, fy)\right] + \delta\frac{d(gx, fx)\,d(gy, fy)}{d(gx, gy)} + Ld(gx, fy) \quad (3.13)$$

For $\alpha, \beta, \lambda, \delta, L$ 0 such that $\alpha + 2\beta + 2\lambda + \delta + L < 1$.

Applications

Application to electric circuit equation: In this section, we apply our main result to solve the electric circuit equation which is in the form of a second-order differential equation. The result obtained from the study conducted by Tomar and Sharma in 2018, showed the coincidence and common fixed-point respecting F-contraction. Hence, by using their theorems, the problems of electric circuit equations which are second-order differential equations were solved. Figure 4 depicts a second-order circuit with a series connection of a resistor R, an inductor L, a capacitor C, and a Voltage V source which contains an electromotive force E (supplied by a battery or generator).

Current I, is the time rate t, of change of charge q, past a given point. However, q is the intrinsic property of matter responsible for electric phenomena. Thus, I can be expressed as;

$$I = \frac{dq}{dt}$$

Hence, the voltage drops in each component are thus;

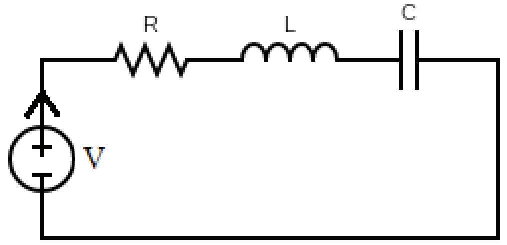

Figure 4: (A series *RLC* circuit).

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(i) $V = IR$;

(ii) $V = \frac{q}{C}$;

(iii) $V = L\frac{dI}{dt}$

For V_R, V_G, V_L are the voltage drops in the resistor, capacitor, and inductor.

From Kirchhoff's voltage law (KVL); the sum of the voltage sources around any closed path, equals the sum of the potential drops around that path.

Using equations (i-iii), KVL can be implemented as follows;

$$IR + \frac{q}{c} + L\frac{dI}{dt} = V(t),$$

or

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = V(t),$$

$$q(0) = 0, q'(0) = 0. \tag{4.1}$$

Where $V(t)$ is the input to the circuit, the output of the circuit, likewise, called the response of the circuit, can be the voltage or current of any device in the circuit. This output is frequently referred to as the voltage of a capacitor or the current of an inductor.

The Green function associated with (4.1) is given by

$$G(t,s) = \begin{cases} -se^{\tau(s-t)}, & 0 \le s \le t \le 1 \\ -te^{\tau(s-t)}, & 0 \le t \le s \le 1, \end{cases} \tag{4.2}$$

Where $\tau > 0$ is a constant, calculated in terms of R and L. Let $X = C([0, a])$, R^+ be the set of all non-negative real-valued functions defined on $[0, a]$.

For an arbitrary $u \in X$, we define

$$u_I = sup_{t\in[0,a]}\left\{x(t)e^{-2\tau t}\right\}. \tag{4.3}$$

Define d: $X \times X \to R^+$ by

$$d(u,v) = u - v_\tau = sup_{t\in[0,a]}\left\{\left|u(t) - v(t)\right|e^{-2\tau t}\right\}. \tag{4.4}$$

Then, clearly (X, d) is a metric space.

We now state and prove the result for the existence of a solution of the LCR-circuit equation of the second-order differential equation:

Theorem 4.1. Let f, g: $C([0, a]) \to C([0, a])$ be self-maps of a metric space (X, d) such that the following conditions hold:

(i) there exists a function K: $[0, a] \times [0, a] \times R \to R$ such that

$$K(t,s,u) - K(t,s,v) \le \tau^2 e^{-\tau} M(u,v),$$

Where

$$M(gu,gv) = max\left\{ \begin{array}{c} d(gu,gv), d(gu,fv), d(gv,fu), d(gu,fu), d(gv,fv), \frac{d(gu,fu)d(gv,fv)}{d(gu,gv)}, \\ \frac{d(gu,fu)d(gv,fv)}{d(gu,gv)+d(gu,fv)+d(gv,fu)}, \frac{d(gu,fu)d(gu,fv)+d(gv,fu)d(gv,fv)}{d(gv,fu)+d(gu,fv)} \end{array} \right\},$$

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For $F \in \mathbb{F}$, $\tau \in \mathbb{R}^+$, $t, s \in [0,a]$ and $u, v \in \mathbb{R}^+$;

(ii) $\lim_n fx_n = t = \lim_n gx_n$ for some $t \in C[(o,a)]$, there exists a sequence $\{y_n\}$ satisfying $\lim_n fy_n = u = \lim_n gy_n$ for some $u \in C[(o,a)]$ such that $\lim_n fgy_n = fu, \lim_n gfy_n = gu$ and $\lim_n fgy_n = \lim_n gfy_n$.

(iii) for all $x \in X$, $fx = gx$ implies $ffx = gfx$.

Then, equation (4.1) has a solution.

Proof Above problem is equivalent to the integral equation

$$u(t) = \int_0^t G(t,s) K(t,s,u(s)) ds,\ t,s \in [0,a]. \tag{4.5}$$

Consider the self-maps $f : X \to X$, defined by

$$fx(t) = \int_0^t G(t,s) K(t,s,u(s)) ds,\ t \in [0,a], a > 0. \tag{4.6}$$

Then clearly u^* is a solution of (4.5), if and only if u^* is a common fixed point of f and g. From (i), for all $u, v \in X$, we have

$$|fu(t) - fv(t)|$$

$$\leq \int_0^t G(t,s) |K(t,s,u(s)) - K(t,s,v(s))| ds$$

$$\leq \int_0^t G(t,s) \tau^2 e^{-\tau} M(u,v) ds$$

$$|fu(t) - fv(t)| \leq \int_0^t \tau^2 e^{-\tau} e^{2\tau s} e^{-2\tau s} M(u,v) G(t,s) ds$$

$$\leq \tau^2 e^{-\tau} M(u,v)_\tau \times \int_0^t e^{2\tau s} G(t,s) ds$$

So, we get

$$\leq \tau^2 e^{-\tau} M(u,v)_\tau \times \left[-\frac{e^{2\tau t}}{r^2} \left(2\tau t - \tau t e^{-\tau t} + e^{-\tau t} - 1 \right) \right]$$

$$|fu(t) - fv(t)| e^{-2\tau t} \leq e^{-\tau} M(u,v)_\tau \times \left[\left(1 - 2\tau t + \tau t e^{-\tau t} - e^{-\tau t} \right) \right]$$

Which implies that

$$|fu(t) - fv(t)| \leq e^{-\tau} M(u,v)_\tau \times \left[\left(1 - 2\tau t + \tau t e^{-\tau t} - e^{-\tau t} \right) \right]$$

From $\left(1 - 2\tau t + \tau t e^{-\tau t} - e^{-\tau t} \right) \leq 1$, we get

$$|fu(t) - fv(t)| e^{-\tau t} \leq e^{-\tau} M(u,v)_\tau$$

or

$$d(fu, fv) \leq e^{-\tau} M(u,v)_\tau.$$

Taking $F(t) = \text{In}(t)$, which is $F \in \mathbb{F}$, we obtain

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$$\ln\left(d\left(fu, fv\right)\right) \leq \ln\left[e^{-\tau} M\left(u,v\right)_{\tau}\right].$$

i.e.,

$$\tau + \ln\left(d\left(fu, fv\right)\right) \leq \ln M\left(u,v\right)_{\tau}.$$

Clearly using (ii) and (iii), all conditions of Theorem 3.3 are satisfied by operators f and g taking $F(x) = \text{In } x$. Hence f and g have a common fixed point which is the solution of the differential equation arising in the electric circuit equation.

Application to Volterra type integral equation for Ciri´c type F-contraction: Next, we discuss the application of Theorem 3.3 to the Volterra-type integral equation as follows:

$$u\left(t\right) = \int_0^t K\left(t,s,u\left(s\right)ds + h\left(t\right)\right); \tag{4.7}$$

For $t, s \in [0, a]$, where $a > 0$. Let $X = C([0, a], R)$ be the space of all functions defined on $[0, a]$. For $u \in C([0, a], R)$, define supremum norm as:

$$u = sup_{t\in[0,a]}\left\{u\left(t\right)e^{-\tau t}\right\},$$

Where $\tau > 0$ is arbitrary. Let (X, d) be a metric space endowed with the metric

$$d\left(u,v\right) = sup_{t\in[0,a]}\left\{\left|u\left(t\right) - v\left(t\right)\right| e^{-\tau t}\right\}, \tag{4.8}$$

for all $u, v \in C([0, a], R)$.

Theorem 4.2 Let $f, g: C([0, a]) \to C([0, a])$ be self-maps of a metric space (X, d) such that the following conditions hold:

(i) there exists a function $K: [0, a] \times [0, a] \times R \to R$ such that

$$K\left(t,s,u\right) - K\left(t,s,v\right) \leq \tau e^{-\tau}\left|M\left(u,v\right)\right|,$$

Where $M(gu, gv) =$

$$M\left(gu,gv\right) = max\left\{\begin{array}{l} d\left(gu,gv\right), d\left(gu,fv\right), d\left(gv,fu\right), d\left(gu,fu\right), d\left(gv,fv\right), \dfrac{d\left(gu,fu\right)d\left(gv,fv\right)}{d\left(gu,gv\right)}, \\ \dfrac{d\left(gu,fu\right)d\left(gv,fv\right)}{d\left(gu,gv\right)+d\left(gu,fv\right)+d\left(gv,fu\right)}, \dfrac{d\left(gu,fu\right)d\left(gu,fv\right)+d\left(gv,fu\right)d\left(gv,fv\right)}{d\left(gv,fu\right)+d\left(gu,fv\right)} \end{array}\right\},$$

For $F \in F$, $\tau \in R^+$, $t, s \in C([0, a])$ and $u, v \in C([0, a]), R)$;

(ii) $\lim_n fx_n = t = \lim_n gx_n$ for some $t \in C([0, a])$, there exists a sequence $\{y_n\}$ satisfying $\lim_n fy_n = u = \lim_n gy_n$ for some $t \in C([0, a])$ such that $\lim_n fgy_n = fu, \lim_n gfy_n = gu$ and $\lim_n fgy_n = \lim_n gfy_n$.

(iii) for all $x \in X$, $fx = gx$ implies $ffx = gfx$.

Then the integral equation (4.7) has a solution in X.

Proof Let

$$fu\left(t\right) = \int_0^t K\left(t,s,u\left(s\right)ds + h\left(t\right)\right), t \in \left[0,a\right], a > 0.$$

From (ii), we have

$$\left|fu\left(t\right) - fv\left(t\right)\right|$$

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$$= \int_0^t \left| K\left(t,s,u\left(s\right)\right) - K\left(t,s,v\left(s\right)\right) \right| ds$$

$$\leq \int_0^t \tau e^{-\tau} \left| M\left(u,v\right) e^{-\tau s} \right| e^{\tau s} ds$$

$$\leq \int_0^t \tau e^{-\tau} M\left(u,v\right)_\tau e^{\tau s} ds$$

$$\leq \tau e^{-\tau} M\left(u,v\right)_\tau \int_0^t e^{\tau s} ds$$

$$\leq \tau e^{-\tau} M\left(u,v\right)_\tau \frac{1}{\tau} . e^{\tau t}$$

$$\leq e^{-\tau} M\left(u,v\right)_\tau e^{\tau t}.$$

Which implies that

$$\left| fu\left(t\right) - fv\left(t\right) \right| e^{-\tau t} \leq e^{-\tau} M\left(u,v\right)_\tau$$

or

$$fu\left(t\right) - fv\left(t\right)_\tau \leq e^{-\tau} M\left(u,v\right)_\tau .$$

Taking $F(t) = \text{In}(t)$, which is $F \in F$, we obtain

$$\tau + \ln fu\left(t\right) - fv\left(t\right)_\tau \leq \ln M\left(u,v\right)_\tau .$$

Using (ii) and (iii), all the conditions of Theorem 3.3 are satisfied for $F(x) = \text{In}(x)$.

Hence, integral equations given in (4.7) have a unique solution.

Application to Volterra type integral equation for Hardy-Rogers -type F-contraction: Now, we solve the Volterra-type integral equation using Hardy-Rogers-type F-contraction.

Define supremum norm as:

$u = sup_{t\in\left[0,a\right]} \left\{u\left(t\right) e^{-\tau t}\right\}$, where $\tau > 0$ is arbitrary.

Let $X = C([0, a])$ and (X, d) be the metric space of all real-valued functions endowed with the metric $d\left(u,v\right) = sup_{t\in\left[0,a\right]} \left\{\left|u\left(t\right) - v\left(t\right)\right| e^{-\tau t}\right\}$.

Consider an integral equation

$$u\left(t\right) = \int_0^t K\left(t,s,u\left(s\right) ds + h\left(t\right)\right), t \in \left[0,a\right], a > 0. \quad (4.9)$$

Theorem 4.3. Let f, g: $C([0, a]) \to C([0, a])$ be self-maps of a metric space (X, d) such that

(i) there exists a function K: $[0, a] \times [0, a] \times R \to R$ and $\tau >$ 0such that

$$K\left(t,s,u\right) - K\left(t,s,v\right) \leq \frac{e^{-\tau}}{\alpha\left(u+v\right)} \left|u - v\right|,$$

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$$\left|\int_0^t \frac{e^{\tau s}}{\alpha\left(u(s)+v(s)\right)}ds\right| \le e^{\tau t},$$ for all $t, s \in [o, a]$ and $u, v \in X$;

(ii) $\lim_n fx_n = t = \lim_n gx_n$ for some $t \in C([0, a])$, there exists a sequence $\{y_n\}$ satisfying $\lim_n fy_n = u = \lim_n gy_n$ for some $u \in C([o, a])$ such that $\lim_n fgy_n = fu$, $\lim_n gfy_n = gu$ and $\lim_n fgy_n = \lim_n gfy_n$.

(iii) $e^{-\tau s}\left|u(s)-v(s)\right| = M(u,v)$,

where

$$M(gx,gy) = \begin{array}{l}\alpha d(gx,gy)+\beta\left[d(gx,fy)+d(gy,fx)\right]+\gamma\left[d(gx,fx)+d(gy,fy)\right]+\delta\frac{d(gx,fx)d(gy,fy)}{d(gx,gy)}+\\ \lambda\frac{d(gx,fx)d(gy,fy)}{d(gx,gy)+d(gx,fy)+d(gy,fx)}+\mu\frac{d(gx,fx)d(gx,fy)+d(gy,fx)d(gy,fy)}{d(gy,fx)+d(gx,fy)}\end{array} \tag{4.10}$$

for $\alpha,\beta,\gamma,\delta,\lambda,\mu \ge 0, \alpha+2(\beta+\gamma)+\delta+\lambda+\mu<1, F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$,

(iv) for all $x \in X$, $fx = gx$ implies $ffx = gfx$.

Then, the integral equation (4.9) has a solution in X.

Proof Let

$$u(t) = \int_0^t K\left(t,s,u(s)ds+h(t)\right), \tag{4.11}$$

For $t, s \in [0, a]$, h: $[0, a] \to R$ are functions for each $t \in ([0, a])$, $a > 0$. First, we show that f is a Hardy-Rogers-type F-contraction.

$$\left|fu(t)-fv(t)\right|$$

$$= \int_0^t \left|K\left(t,s,u(s)\right)-K\left(t,s,v(s)\right)\right|ds$$

$$\le \int_0^t \frac{e^{-\tau}}{\alpha\left(u(s)+v(s)\right)}\left|u(s)-v(s)\right|ds$$

$$\le \int_0^t \frac{e^{-\tau}e^{\tau s}}{\alpha\left(u(s)+v(s)\right)}\left|u(s)-v(s)\right|e^{-\tau s}ds$$

$$\le e^{-\tau}M(u,v)_\tau \int_0^t \frac{e^{\tau s}}{\alpha\left(u(s)+v(s)\right)}ds$$

$$\le e^{\tau t}e^{-\tau}M(u,v)_\tau .$$

Which implies that

$$\left|fu(t)-fv(t)\right|e^{-\tau t} \le e^{-\tau}M(u,v)_\tau$$

or

$$d(fu,fv) \le e^{-\tau}M(u,v)_\tau .$$

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Taking $F(t)$ = In(t), which is $F \in F$, we obtain

$$\ln\left(d\left(fu, fv\right)\right) \le \ln e^{-\tau} M\left(u,v\right)_{\tau}.$$

i.e.,

$$\tau + \ln\left(d\left(fu, fv\right)\right) \le \ln M\left(u,v\right)_{\tau}.$$

Thus f is an F-contraction of Hardy-Rogers-type with $\alpha < 1$, $\beta = \gamma = \delta = \lambda = \mu = 0$, and F($x$) = In x. Using (ii) and (iii) all other conditions of Theorem 3.10 immediately hold. Therefore, the operators f and g have a common fixed point, i.e., the integral equation (4.9) has a solution in X.

Remark: The presented results can be extended to other contraction mappings with metric type.

Conclusion

In this article, we study coincidence fixed point and common fixed point of discontinuous non-compatible faintly compatible pair of self maps in non-complete metric space without using containment requirement of range space of involved maps satisfying Ciri´c type F-contraction and Hardy-Roger type F-contraction. Some illustrative examples associated with pictographic validations to demonstrate the main results and to show the genuineness of our results are provided. We also consider the application of our results to the study of the two-point boundary value problem related to the second-order differential equation, solving the two-point boundary value problem of the second-order differential equation arising in the electric circuit equation. Further Volterra type integral equation is solved using Ciri´c type F-contraction as well as Hardy Roger type F-contraction.

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