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Research Article

Analyzing twin primes, Goldbach's strong conjecture and Polignac's conjecture

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Abstract

Here we analyze three well-known conjectures: (i) the existence of infinitely many twin primes, (ii) Goldbach's strong conjecture, and (iii) Polignac's conjecture. We show that the three conjectures are related to each other. In particular, we see that in analysing the validity of Goldbach's strong conjecture, one must consider also the existence of an infinite number of twin primes. As a consequence of how we approach this analysis, we also observe that if this conjecture is true, then so is Polignac's conjecture. Our first step is an analysis of the existence of infinitely many twin prime numbers. For this, using the formula $4((n-1)!+1) \equiv -n \pmod{n(n+2)}$ – satisfied if and only if $(n, n+2)$ are twin primes –, together with Wilson's theorem, we obtain conditions that must be met for two numbers to be twin primes. Our results, obtained from an analytic and functional study, lead us to conclude that there may exist infinitely many twin primes. Next, we consider the validity of Goldbach's strong conjecture. After showing that the conjecture is true for the first even numbers, we notice a pattern that we analyze for any even number, reducing it to three cases: (i) when the even number $2n$ is two times a prime number n; (ii) when the even number $2n$ is such that $n=2m$, with $2m-1$ and $2m+1$ twin primes; (iii) all other cases, i.e., for any $2n$ even number $\forall n \in \mathbb{N}$ with $n > 1$, n prime or not, with independence of $n=2m$ being $2m-1$ and $2m+1$ twin primes or not. In this last case, we show that one can always find a certain $r \in \mathbb{N}$ such that $1 < r < n$ satisfying that $n - r$ and $n + r$ are primes, so that their sum is $2n$. In this case, we use the reduction to absurd method, and our results lead us to conclude that Goldbach's strong conjecture is true to the best of our calculations, and Polignac's conjecture as well.

Introduction and background

In this paper we analyze three well-known mathematical conjectures: the existence of infinite Twin Prime numbers, Goldbach's Strong Conjecture, and Polignac's Conjecture.

The Goldbach Conjecture, proposing that every even number greater than 2 can be expressed as the sum of two prime numbers, alongside the concepts of symmetry and the existence of twin prime numbers, shares a profound underlying symmetry that also connects with Polignac's Conjecture. This intricate relationship highlights an elegant symmetry in the realm of prime numbers, suggesting that the distribution and pairing of primes (as seen in twin primes and the pairs fulfilling Goldbach's strong conjecture) are not random but follow a pattern that might be fundamentally linked to Polignac's Conjecture. The interconnectedness of these conjectures and the concept of symmetry point to a deeper, perhaps inherent, order and balance within the structure of the mathematical universe.

Although it is not usual, and not normally expected for three conjectures so relevant, with so much historical value, and said independently by different mathematicians (Christian Goldbach in 1742, Alphonse de Polignac in 1826), sometimes it can occur that different events are related to each other. This can be either because for A to happen it is necessary for B and C and D to happen (If B and C and D are true then is A true), or simply because one event can be a direct consequence of another (If A is true then is B true), and this is our case.

Specifically, and as this analysis is approached, it is observed that in order to be able to analyze the validity of Goldbach's Strong Conjecture, one of the cases to be analyzed is nothing more nor less than the validity of the existence of infinite twin primes. As a consequence of the way in which the analysis of Goldbach's Strong Conjecture is approached, it is observed that if this conjecture is true, then Polignac's conjecture is also true.

For this reason, similar to the way in which C. Jouis in 2013 created the Neo-topology from the definition of a ``thickness border'' in a set A to measure the appropriate degree of typicality [1,2], in our case, and even though we will not use topological concepts for this analysis, we believe it is appropriate to categorize the three conjectures with a common property and analyze them together in a single article. Note that all three conjectures state that the number 2, or an even number greater than 2, can be written as the subtraction (Twin primes or Polignac's Conjecture) or the sum (Goldbach's Strong Conjecture) of two prime numbers.

Twin primes and Goldbach's conjecture have also deep implications in other fields of science. For instance, in Physics, it is well known that certain quantum states can codify interesting functions of prime numbers [3] and that these can also be computed with quantum computers [4]. In fact, the dynamics of entangled quantum systems have also been used to identify prime numbers [5]. Goldbach's conjecture has also proven relevant in high-energy physics [6], and in the calculation of expectation values of the number operator in Fock states [7].

Additionally, prime numbers and their associated theories and conjectures collectively form a kind of ontology, a fixed framework where the hierarchy of classes and their interrelations remain constant. These ontologies map out conceptual realms that are timeless and unchanging, with no alteration in class hierarchy or the dynamics between different classes. Changes within this structure are limited to the emergence, extinction, or modification of properties of individual entities. A recent example illustrating this concept comes from paleontology and systematics, which deal with cataloging and organizing species diversity across time. Here, the challenge arises from evolution, as the traditional static network of class relationships must now incorporate a temporal aspect to accurately reflect evolutionary changes. To determine if a subclass is non-typical, concepts from topology such as the interior, exterior, boundary, and closure are utilized. This allows for the representation of atypical entities within ontologies by establishing a network of inclusion and belonging relationships, drawing upon classical topological operations [8].

Let us then begin by introducing Twin Prime numbers.

It is well known in mathematics, and more specifically in number theory, that two prime numbers (p, q) are *twin primes numbers* if being $q > p$, then $q = p + 2$.

Hence, primes numbers 3 and 5 form a pair of twin primes. Other examples are (5, 7), (11, 13), (29, 31), etc.

The twin primes conjecture states *the existence of infinite pairs of twin primes.*

The majority of mathematicians hold the view that the conjecture is valid, and grounded in empirical data and inferential arguments regarding the probabilistic patterns of prime numbers.

In 1849, de Polignac posited a broader hypothesis stating that for each natural number k, an infinite number of prime numbers p exist for which $p+2k$ is prime as well [9]. The special scenario where $k=1$ within de Polignac's conjecture directly corresponds to the twin prime conjecture.

The question of whether an infinite number of twin primes exist or if a maximum pair exists remains unanswered. Yitang Zhang's breakthrough in 2013, along with contributions from James Maynard, Terence Tao, and several others, has significantly advanced the effort to demonstrate the infinite nature of twin primes. However, as of now, this puzzle is still fundamentally unresolved [10].

On April 17, 2013, Yitang Zhang presented proof demonstrating that there exists an integer N, smaller than 70 million, for which an infinite number of prime pairs exist that have a difference of N [11]. Zhang's paper gained acceptance by the Annals of Mathematics in the early days of May 2013 [12]. Following this, Terence Tao suggested a collective endeavor under the Polymath Project to refine Zhang's initial bound [13]. By April 14, 2014, exactly one year after Zhang's original declaration, this bound had been narrowed down to 246 [14].

Some important properties to mention are:

- Every twin primes pair except (3,5) is of the form $(6n - 1, 6n + 1)$ for some natural number n; that is, the number between the two primes is a multiple of 6 [15].
- Brun's theorem: Viggo Brun showed in 1915 that the sum of reciprocals of the twin primes was convergent [16].
- It has been proven that the pair $(n, n+2)$ is a pair of twin primes if and only if $4((n - 1)! + 1) \equiv -n \pmod{n(n + 2)}$.

The last property is a direct consequence of Wilson's theorem applied to n and $n+2$ if they both are prime numbers. Wilson's theorem is an old theorem stated for the first time by Ibn al-Haytham (c. 1000 AD) [17]. Many years later, in 1770 it was found by John Wilson [18] and Edward Waring announced the theorem, although neither he nor Wilson could prove it. It was Lagrange who gave the first proof in 1771 [19], and it seems that Leibniz was also aware of this result a century earlier, but he never published it [20].

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In addition to the references cited above, it is also important to cite some of the most recent works published on the topic. For example, Marc Wolf, François Wolf, and François-Xavier Villemin studied in 2019 the distribution of consecutive composite odd numbers and twin primes [21], in 2020 Berndt Gensel for twin primes greater or equal to 4.5 presented a new approach to prove the Twin Prime Conjecture by a sieve method to extract all Twin Primes on the level of the Twin Prime Generators [22], and in 2021 Madieyna Diouf also presented a study to analyze the distribution of twin primes numbers [23].

As we have mentioned before, the Twin Primes Conjecture is related to Goldbach´s Strong Conjecture, because being more precise, the existence of infinite twin prime numbers allows us to ensure that there exist infinite even numbers $2n$ such that $2n - 1$ and $2n+1$ are prime numbers, twin prime numbers. Hence, this will imply that an infinite number of even numbers can be written as the sum of two twin prime numbers, and we will have proved one important part of Goldbach´s Strong Conjecture.

In this paper we analyze the possibility of the existence of infinite twin prime numbers, or on the contrary, if we can guarantee that there is a finite number of them. Using the formula $4((n-1)!+1) \equiv -n \pmod{n(n+2)}$ and Wilson's Theorem, we show which conditions are met by twin primes, and what would happen if we assume that there is a finite number of them.

Next, we consider Goldbach's Strong Conjecture, one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states:

``Every even integer greater than 2 can be expressed as the sum of two primes [24]".

The conjecture has been shown for all integers less than 4×10^{18} [25].

On 7th June 1742, Christian Goldbach wrote a letter to Leonhard Euler [26] in which he proposed the following conjecture:

``Every integer which can be written as the sum of two primes, can also be written as the sum of as many primes as one wishes, until all terms are units".

He then proposed a second conjecture (the weak conjecture) in the margin of his letter:

``Every integer greater than 5 can be written as the sum of three primes"

Euler replied and reminded Goldbach of an earlier conversation they had, in which Goldbach remarked his original conjecture followed by the following statement:

``Every even integer greater than 2 can be written as the sum of two primes".

Which is actually known as Goldbach's strong conjecture.

Many mathematicians have obtained important results about both conjectures.

Using Vinogradov's method, Chudakov [27], Van der Corput [28] and Estermann [29] proved that almost all even numbers can be written as the sum of two primes.

Lev Schnirelmann [30,31] established that any natural number greater than 1 can be expressed as the sum of no more than a specific constant c number of prime numbers, a constant that can be calculated. This discovery has been refined by several researchers, including Olivier Ramaré, with the most significant improvement arising from Harald Helfgott's proof of the weak Goldbach conjecture [32]. Helfgott's work implies that every even number $n \geq 4$ can be represented by the sum of at most four prime numbers.

In 1973, Chen Jingrun utilized sieve theory techniques to demonstrate that every sufficiently large even number can be decomposed into the sum of two primes or a prime and a semiprime [33].

Hugh Montgomery and Robert Charles Vaughan revealed that the majority of even numbers can be written as the sum of two prime numbers.

Linnik identified a constant k indicating that every sufficiently large even number could be the sum of two primes and no more than k powers of 2. Roger Heath-Brown and Jan-Christoph Schlage-Puchta determined that a value of $k=13$ is applicable [32], which was later improved to $k=8$ by Pintz and Ruzsa [34].

Harald Helfgott [35,36] did considerable work on Goldbach's weak conjecture, culminating in 2013, the weak conjecture fully proved for all odd integers greater than 7.

In addition to the references cited above, it is also important to cite some of the most recent works published about this. For example, Andrei-Lucian Drăgoi published an article about a "vertical" generalization of the binary Goldbach's Conjecture as applied on primes with prime indexes of any order i (i-primes) [37], and Salman Mahmud in 2020 demonstrated a probabilistic heuristic justification for the Goldbach's strong conjecture [38].

In this article, we analyze it for any even number using for this purpose the study of 3 cases. The first case is when the even number $2n$ is two times a prime number n. The second case is when the even number $2n$ is such that $n = 2m$ and $2m - 1$ and $2m + 1$ are primes numbers (more exactly twin prime numbers). Third case, in general $\forall n \in \mathrm{N}$ when $n > 1$ is prime or not and with independence of the case $n = 2m$ being $2m - 1$ and $2m + 1$ twin primes or not. In this last case, we can always find a certain $r \in \mathbb{N}$ such that $1 < r < n$ satisfying that $n - r$ and $n + r$ are prime numbers, and adding them of course we obtain $2n$. To analyze the general case we use the reduction to absurd method, and the results obtained lead us to conclude that Goldbach's Strong Conjecture is true to the best of our calculations, as well as Polignac's Conjecture.

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The paper is organized as follows: in Sec.2 we show if it is possible or not that there exist infinitely many twin prime numbers. In Sec.3 we analyze the validity of Goldbach's Strong Conjecture and as a consequence of this result, we analyze also the validity of Polignac's Conjecture. Finally, in Sec.4 we present our conclusions.

Is there an infinite number of twin primes?

In this section, we analyze if the number of twin primes could be infinite or not.

First, using one property mentioned in the Introduction, we know that:

$(n,n + 2)$ are twin prime numbers if and only if $4((n - 1)! + 1) \equiv -n \pmod{n(n + 2)}$.

That is, if and only if $\exists A \in \mathbb{N}$ such that:

$$4((n-1)!+1)+n = An(n+2) \tag{1}$$

The value of A is not bounded, that is, when n increases then A increases. For instance:

- For the pair (3,5), we obtain $A = 1$
- For the pair (5,7), we obtain $A = 3$
- For the pair (11,13), we obtain $A = 101505$
- For the pair (17,19), we obtain $A = 259105757127$
- etc.

That is $A \geq 1$.

Additionally, Wilson's theorem states that if n is a prime number then $(n - 1)! \equiv -1 \pmod n$, that is, $\exists B \in \mathbb{N}$ such that:

$$(n-1)! = Bn - 1 \tag{2}$$

The value of B is not bounded, that is, when n increases then B increases. For instance:

- For the pair (3,5), we obtain $B = 1$
- For the pair (5,7), we obtain $B = 5$
- For the pair (11,13), we obtain $B = 329891$
- For the pair (17,19), we obtain $B = 1230752346353$
- etc.

That is $B \geq 1$.

If we compare the values of A and B, for each pair of twin prime numbers we observe that $B \geq A$ for those twin primes that we have analyzed. And it also happens for all $(n,n + 2)$ twin prime numbers, that is, $A = \frac{4((n-1)!+1)+n}{n(n+2)} \leq \frac{(n-1)!+1}{n} = B$, because:

$(n + 2)((n - 1)! + 1) = (n + 1 + 1)((n - 1)! + 1) = (n + 1)((n - 1)! + 1) + (n - 1)! + 1$

$$\geq 4((n-1)!+1)+n. \tag{3}$$

That is:

$$(n+2)((n-1)!+1) \geq 4((n-1)!+1)+n. \tag{4}$$

Multiplying the above inequality by n, we have:

$$n(n+2)((n-1)!+1) \geq n(4((n-1)!+1)+n). \tag{5}$$

Dividing the above inequality by $n(n + 2)$, we have:

$$(n-1)!+1 \geq \frac{4((n-1)!+1)+n}{n+2}. \tag{6}$$

Dividing the above inequality by n, we have:

$$B = \frac{(n-1)!+1}{n} \geq \frac{4((n-1)!+1)+n}{n(n+2)} = A. \tag{7}$$

That is, B $\geq A$, as we want to prove.

Hence, if we substitute $(n - 1)!$ in Eq.(1) by $Bn-1$, we have:

$$\begin{aligned} 4(Bn-1+1)+n &= An(n+2), \\ 4Bn+n &= An(n+2), \\ 4B+1 &= A(n+2), \\ n+2 &= \frac{4B+1}{A}. \end{aligned} \tag{8}$$

Note that, from the above equation, we get $n = \frac{4B+1-2A}{A}$, and since $n \in \mathbb{N}$, and $n \geq 3$, this implies:

- A is an odd number.
- $4B+1-2A \geq 3A$, that is, $B \geq \frac{5A-1}{4}$

So now, if we consider $n + 2$ as a function of A and B, we will analyze if the function $n+2 \equiv f(A,B) = \frac{4B+1}{A}$ can have a maximum under the constraints, $A \geq 1$ (A odd number), $B \geq 1$, $B \geq A$ and $B \geq \frac{5A-1}{4}$.

First, we analyze the function $z = f(x,y) = \frac{4y+1}{x}$ over the real numbers, and in the feasible region according to the constraints that we have obtained, which can be written in terms of x and y as $x \geq 1$, $y \geq 1$, $y \geq x$ and $y \geq \frac{5x-1}{4}$. We can see what the feasible region looks like in the next (Figure 1).

Then, if we calculate the gradient vector of the function $z = f(x,y) = \frac{4y+1}{x}$, we realize that it is never equal to (0,0) and this implies that this function has neither maximum nor minimum. We can see what this function looks like in the next (Figure 2).

Hence, if we consider this function only over natural numbers and using the above constraints, then over the feasible

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region (not bounded) this function has neither a maximum and only for $(A,B) = (1,1)$ we obtain the minimum value equal to 5. See the graph in the next (Figure 3).

Therefore, we cannot guarantee that there is a finite number of twin primes. On the contrary: these results lead us to conclude that there are infinite twin prime numbers and enable us to approach Goldbach´s Strong Conjecture proof with the knowledge that infinite even numbers exist, which can be expressed as the sum of two twin prime numbers, to the best of our calculations.

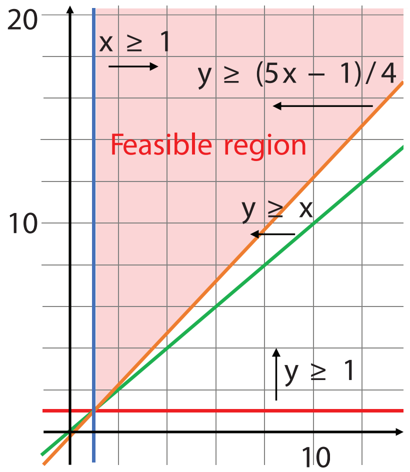

Figure 1: Feasible region of $f(x, y)$ according to the constraints.

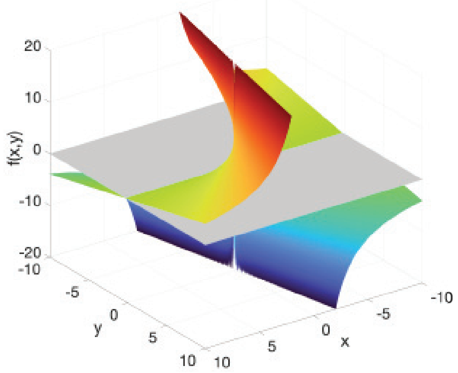

Figure 2: Function $z = f(x, y)$ over the real numbers.

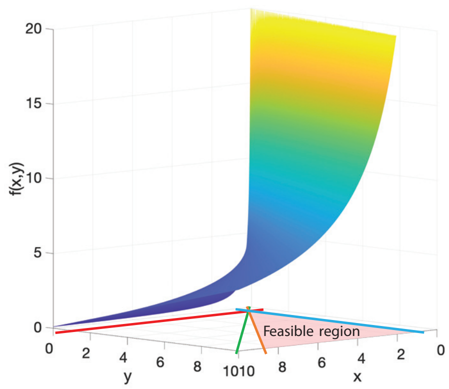

Figure 3: Function $z = f(x, y)$ directly over the feasible region.

Goldbach's Strong Conjecture and Polignac's Conjecture

Let us now use the previous results on twin prime numbers to assess Golbach's (strong) and Polignac's conjectures.

Note that, as we have analyzed in the previous section, there exists an infinite number of even numbers so that they can be written as the sum of two twin prime numbers. This finding is very important because since the twin prime numbers are infinite, this implies that there exist infinite even numbers, such that, they are the result of the sum of two twin prime numbers.

Analysis of the first even numbers

If we analyze Goldbach's strong conjecture for the first even numbers $2n$, we find these prime numbers, and we realize that all of them are of the form $n - r$ and $n + r$ for $r \in \mathbb{N}$.

Let us show this for these first even numbers:

$$\begin{aligned}
&2\cdot 2 = 2+2 \quad (r=0)\\
&2\cdot 3 = 3+3 \quad (r=0)\\
&2\cdot 4 = 3+5 \quad (r=1)\\
&2\cdot 5 = 3+7 = 5+5 \ (r=2, r=0)\\
&2\cdot 6 = 5+7 \quad (r=1)\\
&2\cdot 7 = 3+11 = 7+7 \ (r=4, r=0)\\
&2\cdot 8 = 3+13 = 5+11 \ (r=5, r=3)\\
&2\cdot 9 = 5+13 = 7+11 \ (r=4, r=2)\\
&2\cdot 10 = 3+17 = 7+13 \quad (r=7, r=3)\\
&2\cdot 11 = 3+19 = 5+17 = 11+11 \quad (r=8, r=6, r=0)\\
&2\cdot 12 = 5+19 = 7+17 = 11+13 \quad (r=7, r=5, r=1)\\
&2\cdot 13 = 3+23 = 7+19 = 13+13 \quad (r=10, r=6, r=0)\\
&2\cdot 14 = 5+23 = 11+17 \quad (r=9, r=3)\\
&2\cdot 15 = 7+23 = 11+19 = 13+17 \quad (r=8, r=4, r=2)\\
&2\cdot 16 = 3+29 = 13+19 \quad (r=13, r=3).
\end{aligned} \tag{9}$$

As we can observe, it seems that they follow a pattern.

For instance:

- For the cases 2·2 or 2·3 and all the cases where $2n$ is the result for n a prime number, then the sum of $n+n$ is always valid, and for certain cases like 2·5 not only 5+5=10 is valid, we can observe that adding 3 and 7 (both primes) we obtain also 10.
- There are cases for which n is an even number like 2·6, where 5=6−1 and 7=6+1 are just the numbers above and below number 6, and these are prime numbers, more exactly they are twin primes, such that adding them we obtain in this case 12. Another similar case is 2·12, where 11 and 13 are both twin primes such that the sum is 24, but in this case, there are also other prime numbers giving their sum also equal to 24.
- In general, $\forall n \in \mathbb{N}$, when $n > 1$ is prime or not, or $n-1$ and $n+1$ are twin primes or not, we observe that it seems that we can always find a certain $r \in \mathbb{N}$ such that $1 < r <$

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n satisfying that $n - r$ and $n + r$ are prime numbers, and by adding them we obtain $2n$.

We will analyze in detail these three cases in the next section.

Analysis in general

We need to prove that for $n \geq 2$, $\exists p,q$ both prime numbers, such that $2n = p + q$.

There are two cases where this is satisfied directly. These cases are:

- If n is prime then $2n = n + n$, that is $2n$ can be written as the sum of two primes, both primes being the same n.
- If n is an even number $n = 2m$ for $m \in \mathbb{N}$, such that $2m-1$ and $2m+1$ are twin primes, then $2m - 1 + 2m + 1 = 4m = 2 \cdot 2m = 2n$. That is, $2n = 4m$ is the sum of $2m - 1$ and $2m + 1$, both twin primes, and since there exist infinite twin prime numbers, this implies that there are many cases similar to this.

Additionally, in the previous section we have observed that, in general, $\forall n \in \mathbb{N}$ when n is prime or not, and with independence of the case $n = 2m$ being $2m - 1$ and $2m + 1$ twin primes or not, we observe that $\exists r \in \mathbb{N}$ such that $1 < r < n$ satisfying that $n - r$ and $n + r$ are prime numbers and adding them we obtain $2n$. Therefore, we analyze this case with respect to what follows.

We will assume $n > 3$ because for $n = 2$ or $n = 3$, $2 \cdot 2 = 2 + 2$ and $2 \cdot 3 = 3 + 3$, that is, 2 and 3 are the prime numbers, and also that there are no other prime numbers such that adding them the result will be 4 or 6.

We will use also the reduction to the absurd method.

Then let us assume that it is not true, the statement $\forall n \in \mathbb{N}\ n > 3\ \exists r \in \mathbb{N}\ 1 < r < n$ such that $n - r$ and $n + r$ are prime numbers.

Applying the logic rules to negate a quantifier and the corresponding Morgan law to negate a conjunction, we must then analyze if it can be true or not that $\exists n \in \mathbb{N}\ n > 3$ such that $\forall r \in \mathbb{N}\ 1 < r < n$, $n - r$ or $n + r$ are not prime numbers.

Hence, let us assume that this $n{=}m$, then If $m - r$ is not prime $\forall r \in \mathbb{N}\ 1 < r < n$, then this implies that there are not any prime numbers less than $m-1$, but then this is only possible if $m = 3$, however, we were assuming $n > 3$. Therefore this case cannot be possible.

If $m+r$ is not prime $\forall r \in \mathbb{N}\ \ 1 < r < n$, this implies that there are no prime numbers greater than $m+1$, that is, there would exist a finite number of prime numbers. However, this is not true because there exist infinite prime numbers. Therefore this case is not possible.

Hence, we arrive at a contradiction, and therefore it is satisfied that $\forall n \in \mathbb{N}\ \ n > 3\ \ \exists r \in \mathbb{N}\ \ 1 < r < n$ such that $n-r$ and $n+r$ are prime numbers. Then for all even numbers $2n$ we can write this as the sum of these prime numbers $n-r$ and $n+r$.

Additionally, to prove this last case in which we are saying that the primes p and q are obtained by adding and subtracting a certain r to n, we could also prove it in the following way:

If $2n$ is such that n is a prime number or not, and there exist two prime numbers p and q such that $2n = p + q$ for $p \neq n$ and $q \neq n$, then p and q must be less than $2n$.

Hence, let us assume that $p = 2n - t$ for $t \in \mathbb{N}\ \ t < 2n$. Then, since $p \neq n$ and $q \neq n$, this implies $t \neq n$ and then $t < n$ or $t > n$.

Let us assume that $t < n$, then $p = 2n-t > n$, therefore, since $p > n$ this implies that $p = n+r$ for any $r \in \mathbb{N}\ \ r < n$, and q should be less than n, because if $q > n$ then $p + q > n + n = 2n$. However, we were assuming that $2n = p + q$, therefore q must be less than n, that is, $q = n - s$ for any $s \in \mathbb{N}\ \ s < n$.

Then, since $2n = p+q$ this implies that $2n = n + r + n - s = 2n + r - s$. Then $r - s = 0$, that is, $r = s$ and this implies that the primes p and q are obtained adding and subtracting a certain $r \in \mathbb{N}\ \ r < n$, so that $p = n + r$ and $q = n - r$.

A possible question now could be how to compute these r values. In this sense, since a prime number is an odd number, this implies that if n is an odd number then r must be an even number, and if n is an even number then r must be an odd number.

In addition, and applying Wilson's theorem, which states that n is a prime number if and only if $(n-1)! \equiv -1 \pmod{n}$, then as $n-r$ and $n+r$ must be prime numbers, then the possible values of r must satisfy the next two congruence equations:

$$\begin{aligned}(n-r-1)! &\equiv -1 \,(\mathrm{mod}\ n-r),\\ (n+r-1)! &\equiv -1 \,(\mathrm{mod}\ n+r).\end{aligned} \tag{10}$$

This system of congruence equations can be solved by writing any software code, or for example by using Wolfram Alpha to yield solutions, maybe not for very big natural numbers, but below are results that show the possible r values (r is denoted x in Wolfram) to subtract and add to 7 to obtain 14, or the ones to subtract and add to 8 to obtain16 and the ones to subtract and add to 9001 to obtain 18002 (Figure 4).

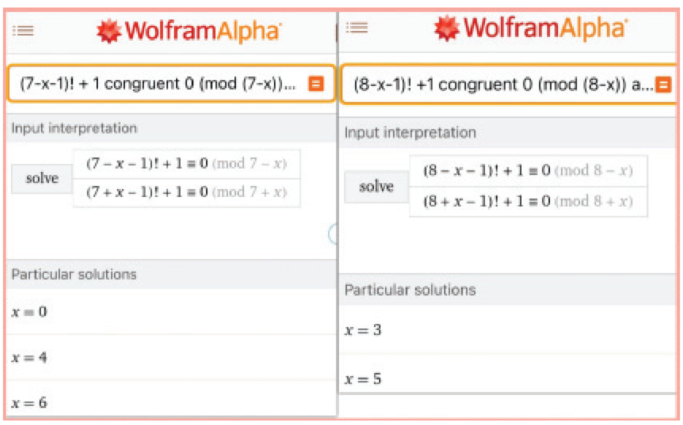

Figure 4: Solutions of $x(r)$ to obtain 14 and 16.

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Note that for the case of 14, r = 0 is computed because 7 is a prime number and 7+7=14, which is correct, as well as the value r = 6 is computed because 1 and 13 are satisfying the equations, however, we discard the value r = 6 because 1 is not a prime number (Figures 5-7).

Therefore, the results lead us to conclude that the Goldbach Strong Conjecture is true, to the best of our observations.

Additionally, with the last result, it is also proved that there are infinitely many primes like p and $p+2r$, because we can say that $n - r = p$ and then $n + r = p + 2r$. That is, there exist infinite pairs of primes $(p, p + 2r)$ for $r \in \mathbb{N}$ $r > 1$, which is equivalent to stating that for all n even natural numbers greater than 2 there exist p and q prime numbers, such that $q > p$ and $n = q - p$, which is the well-known Polignac's Conjecture. Therefore, these results lead us to conclude that Polignac's Conjecture is also true, again to the best of our observations.

Corollary: consequence of the validity of Goldbach's Strong Conjecture

Since $\forall n \in \mathbb{N} \quad n > 2$ even number, can be written as the sum of two prime numbers p and q, then $\forall n \in \mathbb{N} \quad n > 5$ odd number, can be written as the sum of three prime numbers:3, p and q.

WolframAlpha

(9001-x-1)! + 1 congruent 0 (mod (9...

Input interpretation

solve $(9001 - x - 1)! + 1 \equiv 0 \pmod{9001 - x}$, $(9001 + x - 1)! + 1 \equiv 0 \pmod{9001 + x}$

Particular solutions	Particular solutions	Particular solutions	Particular solutions	Particular solutions
$x = 0$	$x = 438$	$x = 1068$	$x = 2160$	$x = 3048$
$x = 108$	$x = 462$	$x = 1242$	$x = 2172$	$x = 3162$
$x = 180$	$x = 612$	$x = 1302$	$x = 2238$	$x = 3210$
$x = 198$	$x = 648$	$x = 1320$	$x = 2298$	$x = 3252$
$x = 240$	$x = 732$	$x = 1332$	$x = 2310$	$x = 3300$
$x = 282$	$x = 738$	$x = 1398$	$x = 2328$	$x = 3342$
$x = 402$	$x = 768$	$x = 1428$	$x = 2382$	$x = 3378$
$x = 420$	$x = 780$	$x = 1452$	$x = 2550$	$x = 3420$
	$x = 810$	$x = 1512$	$x = 2700$	$x = 3432$
	$x = 900$	$x = 1650$	$x = 2730$	$x = 3552$
	$x = 948$	$x = 1788$	$x = 2838$	$x = 3582$
	$x = 1008$	$x = 2118$	$x = 2922$	$x = 3588$
	$x = 1038$	$x = 2130$	$x = 2958$	$x = 3720$

Figure 5: Solutions of $x(r)$ to obtain 18002 (picture 1 of 3).

Particular solutions	Particular solutions	Particular solutions	Particular solutions	Particular solutions	Particular solutions
$x = 3822$	$x = 4410$	$x = 5430$	$x = 6288$	$x = 6972$	$x = 7830$
$x = 3888$	$x = 4440$	$x = 5460$	$x = 6312$	$x = 6990$	$x = 7878$
$x = 3978$	$x = 4728$	$x = 5502$	$x = 6318$	$x = 7068$	$x = 7962$
$x = 4002$	$x = 4758$	$x = 5532$	$x = 6330$	$x = 7140$	$x = 7980$
$x = 4008$	$x = 4872$	$x = 5628$	$x = 6450$	$x = 7248$	$x = 7992$
$x = 4032$	$x = 4902$	$x = 5682$	$x = 6618$	$x = 7332$	$x = 8010$
$x = 4092$	$x = 4998$	$x = 5730$	$x = 6660$	$x = 7338$	$x = 8190$
$x = 4098$	$x = 5070$	$x = 5820$	$x = 6732$	$x = 7380$	$x = 8292$
$x = 4170$	$x = 5082$	$x = 5922$	$x = 6822$	$x = 7452$	$x = 8340$
$x = 4218$	$x = 5148$	$x = 5982$	$x = 6858$	$x = 7518$	$x = 8358$
$x = 4338$	$x = 5292$	$x = 6030$	$x = 6888$	$x = 7572$	$x = 8382$
$x = 4380$	$x = 5388$	$x = 6198$	$x = 6912$	$x = 7602$	$x = 8388$
$x = 4398$	$x = 5418$	$x = 6270$	$x = 6918$	$x = 7698$	$x = 8400$

Figure 6: Solutions of $x(r)$ to obtain 18002 (picture 2 of 3).

Particular solutions	Particular solutions
$x = 8430$	$x = 8970$
$x = 8538$	$x = 8988$
$x = 8568$	
$x = 8580$	
$x = 8622$	
$x = 8760$	
$x = 8790$	
$x = 8838$	
$x = 8850$	
$x = 8862$	
$x = 8922$	
$x = 8928$	
$x = 8958$	

Figure 7: Solutions of $x(r)$ to obtain 18002 (picture 3 of 3).

Proof

If n is an odd number greater than 5, then $n = 2(m + 1) + 1$ for $m \in \mathbb{N}$ $m \geq 2$, that is, $n = 2m + 2 + 1 = 2m + 3$.

Since $2m$ is an even number, then there exist p and q prime numbers such that $2m = p + q$.

Therefore:

$$n = 2m + 3 = p + q + 3 \tag{11}$$

As we want to prove.

This consequence proves also the weak conjecture of Goldbach, which states that all odd numbers greater than 5 can be expressed as the sum of three primes but adding something else because one of these primes can always be the number 3.

Conclusion

In this article, we analyzed three well-known mathematical conjectures: the conjecture concerning the existence of infinite Twin Primes, Goldbach's Strong Conjecture, and Polignac's Conjecture.

Although it is not usual, or what one might expect for three conjectures so relevant and with so much historical value (and also said independently and by different mathematicians), we realized that the three of them are related to each other. In particular, and as a consequence of how our analysis is approached, it is observed that in order to be able to analyze the validity of Goldbach's Strong Conjecture, one of the cases to be considered is related to the validity of the existence of infinitely-many twin primes. In addition, it is also observed that if this conjecture is true, then Polignac's conjecture is also true.

Therefore, first, we began analyzing the validity or non-validity of the existence of infinite twin primes.

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To do this we used the property $4((n - 1)! + 1) \equiv -n \pmod{n(n + 2)}$, which we know to be true for any pair of twin primes $(n,n + 2)$, as well as Wilson's Theorem.

In this way, we obtained certain conditions that these twin primes must satisfy, and which in turn, only depend on two natural numbers which we call A and B.

These natural numbers A and B are subject to certain restrictions. By extending them to all the real numbers and drawing the feasible region in the plane, we observed that this feasible region is not bounded.

Additionally, if $(n,n+2)$ is a pair of twin primes, $n+2$ and n can be expressed as a function that depends only on these A and B (we call it $f(A,B)$). That is, it was observed that $n+2 = f(A,B)$.

Consequently, the next question we asked ourselves is whether the said function $f(A,B)$ extended to the whole real plane and within the feasible region has a maximum or not. We observed that this function has no maximum in the real numbers, and therefore not in the natural numbers either.

Therefore, from all the above, only one conclusion is valid, which is the existence of infinitely many twin primes numbers.

Secondly, we followed with the study of the validity of Goldbach's Strong Conjecture.

We initially showed that the conjecture is true for those even numbers $2n$ such that n is a prime number.

Then, we showed that it is also true for those even numbers $2n$ such that n is an even number equal to $2m$ and satifying that $2m-1$ and $2m+1$ are twin primes numbers. Therefore, since there are infinitely many twin prime numbers, this fact is important.

Finally, we analyzed in detail that the conjecture is also true $\forall n \in \mathbb{N}$, $n > 2$ prime or not, and with independence of the case $n = 2m$ being $2m - 1$ and $2m + 1$ twin primes or not. To analyze this general case we used the reduction to absurd method, and the results obtained led us to conclude that Goldbach's Strong Conjecture is true, as well as, Polignac's Conjecture as a direct consequence of the validity of Goldbach's Strong Conjecture, to the best of our observations.

The results presented in this paper show also how different conjectures, formulated independently, are actually related to each other, in the sense that proving one of them implies that the other follows. Our findings are relevant beyond pure mathematics, given the impact of prime numbers in other fields of science, such as physics, computer science, cryptography, and more. Exploring the implications of our results in these fields is an interesting research line that we leave for future works.

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