Peertechz

Annals of Mathematics and Physics

MATHEMATICS AND PHYSICS GROUP

ANNALS OF Mathematics and Physics OPEN ACCESS JOURNAL

ISSN: 2689-7636 DOI: https://dx.doi.org/10.17352/amp

Research Article

Coefficient estimates for a subclass of bi-univalent functions associated with the Salagean differential operator

Mohammad Mehdi Shabani[1], Maryam Yazdi[2] and Saeed Hashemi Sababe[3]*

[1]Faculty of Sciences, Emam Ali University, Tehran, Iran

[2]Young Researchers and Elite Club, Malard Branch, Islamic Azad University, Malard, Iran

[3]Mathematical and Statistical Sciences, University of Alberta, Canada

Received: 25 March, 2024
Accepted: 04 April, 2024
Published: 05 April, 2024

***Corresponding author:** Saeed Hashemi Sababe, Mathematical and Statistical Sciences, University of Alberta, Canada, E-mail: Hashemi_1365@yahoo.com

ORCiD: https://orcid.org/0000-0003-1167-5006

Keywords: Bi-univalent functions; Coefficient estimates; Univalent functions; Salagean differential operator

Abstract

In this paper, we present and examine a novel subset of the function class Σ, which consists of analytic and bi-univalent functions defined in the open unit disk U and connected to the Salagean differential operator. Additionally, we determine estimates for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ functions within this new subclass and enhance some recent findings.

2010 Mathematics Subject Classification. 30C45; 30C50.

1. Introduction

Consider the class of functions A defined as

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n, \tag{1.1}$$

where $f(z)$ is analytic in the open unit disk $\mathbb{U}=z\in C:|z|<1$ Let s be the subset of functions $f\in A$ that are univalent in $\mathbb{U}$.

The Koebe one-quarter theorem [3] states that for every $f\in s$, image of $\mathbb{U}$ under f contains a disk of radius $\frac{1}{4}$. Thus, every $f\in s$ has an inverse f^{-1}, defined as

$$f^{-1}(f(z))=z,\ z\in\mathbb{U},$$

and

$$f(f^{-1}(w))=w,\ \text{for } |w|<r_0(f),\ \text{where } r_0(f)\geq\frac{1}{4},$$

with

$$f^{-1}(w)=g(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\ldots \tag{1.2}$$

A function $f\in A$ is bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ be the set of bi-univalent functions in $\mathbb{U}$ given by [1,6].

Several authors have investigated bounds for various subclasses of biunivalent functions [2-5,7-14]. However, the estimation of the Taylor-Maclaurin coefficients $|a_n|$ for $n\in\frac{\mathbb{N}}{\{1,2\}};\mathbb{N}:=\{1,2,3,\cdots\}$ remains an open problem.

In 1983, Salagean [13] introduced the differential operator $\mathcal{D}^k:\mathcal{A}\to\mathcal{A}$ defined by

$$\mathcal{D}^0f(z)=f(z),$$

$$\mathcal{D}^1f(z)=\mathcal{D}f(z)=zf'(z),$$

Citation: Shabani MM, Yazdi M, Sababe SH (2024) Coefficient estimates for a subclass of bi-univalent functions associated with the Salagean differential operator. Ann Math Phys 7(1): 091-095. DOI: https://dx.doi.org/10.17352/amp.000112

$$\mathcal{D}^k f(z) = \mathcal{D}(\mathcal{D}^{k-1} f(z)) = z(\mathcal{D}^{k-1} f(z))',\ k \in \mathbb{N}.$$

That is, the Salagean differential operator D^k applied to a function $f(z)$ is defined recursively as the derivative of $D^{k-1}f(z)$ multiplied by z. This operator is employed in the study of analytic functions, particularly for estimating coefficients of certain classes of functions. It should be noted that

$$\mathcal{D}^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n,\ k \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}.$$

This paper aims to introduce a new subclass of the function class Σ associated with the Salagean differential operator and derive estimates for the coefficients $|a_2|$ and $|a_3|$ functions within these new subclasses of the function class.

2. The subclass $\mathcal{S}_{\Sigma}^{h,p}$

In this section, we introduce and investigate the general subclass $\mathcal{S}_{\Sigma}^{h,p}$.

Definition 2.1: *Let the functions $h, p : \mathbb{U} \to \mathbb{C}$ be so constrained that*

$$\min\{\mathfrak{Re}(h(z)), \mathfrak{Re}(p(z))\} > 0,\ z \in \mathbb{U}, h(0) = p(0) = 1.$$

Also let the function f, defined by (1.1), be in the analytic function class A. we say that

$$f \in \mathcal{S}_{\Sigma}^{h,p}(k,\lambda),\ k \in \mathbb{N}_0, 0 \le \lambda < 1.$$

if the following conditions are satisfied:

$$f \in \Sigma \quad and \quad \frac{\mathcal{D}^{k+1} f(z)}{(1-\lambda)\mathcal{D}^k f(z) + \lambda \mathcal{D}^{k+1} f(z)} \in h(\mathbb{U}),\ z \in \mathbb{U}, \tag{2.1}$$

and

$$\frac{\mathcal{D}^{k+1} g(w)}{(1-\lambda)\mathcal{D}^k g(w) + \lambda \mathcal{D}^{k+1} g(w)} \in p(\mathbb{U}),\ w \in \mathbb{U}. \tag{2.2}$$

where the function $g(w)$ is given by (1.2).

Remark 2.2: *There are many choices of the functions $h(z)$ and $p(z)$ which would provide interesting subclasses of the analytic function class A.*

1. For $h(z) = p(z) = (\frac{1+z}{1-z})^{\alpha}$ and $k=\lambda=0$, we have $\mathcal{S}_{\Sigma}^{h,p}(0,0) = \mathcal{S}_{\Sigma}^{*}(\alpha)$ and $k=1, \lambda=0$, $\mathcal{S}_{\Sigma}^{h,p}(1,0) = \mathcal{K}_{\Sigma}(\alpha)$ where the classes $\mathcal{S}_{\Sigma}^{*}(\alpha)$ and $\mathcal{K}_{\Sigma}(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α corresponding, was introduced and studied by Brannan and Taha [1].
2. For $h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z}$ and $k=\lambda=0$, we have $\mathcal{S}_{\Sigma}^{h,p}(0,0) = \mathcal{S}_{\Sigma}^{*}(\beta)$ and $k=1, \lambda=0$, $\mathcal{S}_{\Sigma}^{h,p}(1,0) = \mathcal{K}_{\Sigma}(\beta)$ where the classes $\mathcal{S}_{\Sigma}^{*}(\beta)$ and $\mathcal{K}_{\Sigma}(\beta)$ of bi-starlike functions of order β and bi-convex functions of order β corresponding, was introduced and studied by Brannan and Taha [1].
3. For $h(z) = p(z) = (\frac{1+z}{1-z})^{\alpha}$ we have $\mathcal{S}_{\Sigma}^{h,p}(k,\lambda) = \mathcal{S}_{\Sigma}^{k,\lambda}(\alpha)$ and taking $h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z}$, $\mathcal{S}_{\Sigma}^{h,p}(k,\lambda) = \mathcal{S}_{\Sigma}^{k,\lambda}(\beta)$ where the classes $\mathcal{S}_{\Sigma}^{k,\lambda}(\alpha)$ and $\mathcal{S}_{\Sigma}^{k,\lambda}(\beta)$ was introduced and studied by J.Jothibaso [6].

3. Coefficient estimates

For proof of the theorem, we need the following lemma.

Lemma 3.1: *(see [3]). If $p \in P$, then $|c_k| \le 2$ for each k, where p is the family of all functions $p(z)$ analytic in $\mathbb{U}$ for which $\mathfrak{Re}(p(z)) > 0, p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ for $z \in \mathbb{U}$.*

Theorem 3.2: *Let $f(z)$ given by the Taylor Maclaurin series expansion (1.1) be in the class $\mathcal{S}_{\Sigma}^{h,p}(k,\lambda), (0 \le \lambda < 1)$. Then,*

$$|a_2| \le \min\left\{\sqrt{\frac{|h'(0)|^2 + |p'(0)|^2}{2^{2k+1}(1-\lambda)^2}}, \sqrt{\frac{|h''(0)| + |p''(0)|}{2^{2k+2}(\lambda^2 - 1) + 8(1-\lambda)3^k}}\right\}, \tag{3.1}$$

and

$$|a_3| \le \min\{\frac{|h'(0)|^2 + |p'(0)|^2}{2^{2k+1}(1-\lambda)^2} + \frac{|h''(0)| + |p''(0)|}{8(1-\lambda)3^k},$$

$$\frac{|h''(0)| + |p''(0)|}{8(1-\lambda)3^k} + \frac{|h''(0)| + |p''(0)|}{2^{2k+2}(\lambda^2 - 1) + 8(1-\lambda)3^k}\}.$$

Proof. First of all, it follows from the conditions (2.1) and (2.2) that

$$\frac{\mathcal{D}^{k+1} f(z)}{(1-\lambda)\mathcal{D}^k f(z) + \lambda \mathcal{D}^{k+1} f(z)} = h(z),\ z \in \mathbb{U}, \tag{3.2}$$

and

$$\frac{\mathcal{D}^{k+1} g(w)}{(1-\lambda)\mathcal{D}^k g(w) + \lambda \mathcal{D}^{k+1} g(w)} = p(w),\ w \in \mathbb{U}, \tag{3.3}$$

where the function $g(w)$ is given by (1.2). respectively, where $h(z)$ and $p(w)$ satisfy the conditions of Definition (2.1). Furthermore, the functions $h(z)$ and $p(w)$ have the following Taylor-Maclaurin series expansions:

$$h(z) = 1 + h_1 z + h_2 z^2 + \ldots$$

and

$$p(w) = 1 + p_1 w + p_2 w^2 + \ldots$$

Citation: Shabani MM, Yazdi M, Sababe SH (2024) Coefficient estimates for a subclass of bi-univalent functions associated with the Salagean differential operator. Ann Math Phys 7(1): 091-095. DOI: https://dx.doi.org/10.17352/amp.000112

Now, equating the coefficients in (3.2) and (3.3), we get

$$2^k(1-\lambda)a_2 = h_1, \tag{3.4}$$

$$2^{2k}(\lambda^2-1)a_2^2 + 2\cdot 3^k(1-\lambda)a_3 = h_2, \tag{3.5}$$

$$-2^k(1-\lambda)a_2 = p_1, \tag{3.6}$$

$$2\cdot 3^k(1-\lambda)(2a_2^2-a_3) + 2^{2k}(\lambda^2-1)a_2^2 = p_2. \tag{3.7}$$

From (3.4) and (3.6), we obtain

$$h_1 = -p_1,$$

and

$$h_1^2 + p_1^2 = 2^{2k+1}(1-\lambda)^2 a_2^2. \tag{3.8}$$

Also, From (3.5) and (3.7), we find that

$$h_2 + p_2 = 2(2^{2k}(\lambda^2-1) + 2\cdot 3^k(1-\lambda))a_2^2. \tag{3.9}$$

Therefore, we find from the equations (3.8) and (3.9) that

$$|a_2|^2 \le \frac{|h'(0)|^2 + |p'(0)|^2}{2^{2k+1}(1-\lambda)^2},$$

and

$$|a_2|^2 \le \frac{|h''(0)| + |p''(0)|}{2^{2k+2}(\lambda^2-1) + 2^3\cdot 3^k(1-\lambda)},$$

respectively. So we get the desired estimate on the coefficient $|a_2|$ as asserted in (3.1). Next, to find the bound on the coefficient $|a_3|$, we subtract (3.7) from (3.5). We thus get

$$h_2 - p_2 = 4\cdot 3^k(1-\lambda)(a_3 - a_2^2). \tag{3.10}$$

Upon substituting the value of a_2^2 from (3.8) into (3.10), it follows that

$$a_3 = \frac{h_1^2 + p_1^2}{2^{2k+1}(1-\lambda)^2} + \frac{h_2 - p_2}{4\cdot 3^k(1-\lambda)}.$$

We thus find that

$$|a_3| \le \frac{|h'(0)|^2 + |p'(0)|^2}{2^{2k+1}(1-\lambda)^2} + \frac{|h''(0)| + |p''(0)|}{8\cdot 3^k(1-\lambda)}.$$

On the other hand, upon substituting the value of a_2^2 from (3.9) into (3.10), it follows that

$$a_3 = \frac{h_2 - p_2}{4\cdot 3^k(1-\lambda)} + \frac{h_2 + p_2}{4\cdot 3^k(1-\lambda) + 2^{2k+1}(\lambda^2-1)}.$$

Consequently, we have

$$|a_3| \le \frac{|h''(0)| + |p''(0)|}{8\cdot 3^k(1-\lambda)} + \frac{|h''(0)| + |p''(0)|}{8\cdot 3^k(1-\lambda) + 2^{2k+2}(\lambda^2-1)}.$$

This evidently completes the proof of Theorem 3.2.

4. Corollaries and consequences

By setting $h(z) = p(z) = (\frac{1+z}{1-z})^{\alpha}$, (0<α≤1) in Theorem 3.2. we get the following consequence.

Corollary 4.1: *Let the function f(z) given by the Taylor-Maclaurin series expansion (1.1) be in the bi-univalent function class* $\mathcal{S}_{\Sigma}^{h,p}(k,\lambda)$ *, (0<λ1). Then*

$$|a_2| \le \begin{cases} \dfrac{2\alpha}{\sqrt{2^{2k+1}(\lambda^2-1) + 4(1-\lambda)3^k}}, & k=0, \\[2ex] \dfrac{2\alpha}{2^k(1-\lambda)}, & k=1,2,3,\ldots \end{cases}$$

and

$$|a_3| \le \begin{cases} \dfrac{2\alpha^2}{2^{2k}(\lambda^2-1) + (2-2\lambda)3^k} + \dfrac{\alpha^2}{(1-\lambda)3^k}, & k=0, \\[2ex] \dfrac{4\alpha^2}{2^{2k}(1-\lambda)^2} + \dfrac{\alpha^2}{(1-\lambda)3^k}, & k=1,2,3,\ldots \end{cases}$$

Remark 4.2: *Corollary 4.1 is an improvement of the following estimates obtained by J.Jothibaso [6].*

Corollary 4.3: *(see[6]) Let the function f(z) given by the Taylor-Maclaurin series expansion (1) be in the bi-univalent function class* $\mathcal{S}_{\Sigma}^{k,\lambda}(\alpha)$ *. Then*

$$|a_2| \le \frac{2\alpha}{\sqrt{4\alpha(1-\lambda)3^k + [2\alpha(\lambda^2-1) - (\alpha-1)(1-\lambda)^2]2^{2k}}},$$

and

$$|a_3| \le \frac{4\alpha^2}{2^{2k}(1-\lambda)^2} + \frac{\alpha}{3^k(1-\lambda)}.$$

Remark 4.4: *It is easy to see that [(i)]*

1. For the coefficient $|a_2|$, If k = 0 and 0<α≤1, we have

$$\frac{2\alpha}{\sqrt{2^{2k+1}(\lambda^2-1) + 4(1-\lambda)3^k}} \le \frac{2\alpha}{\sqrt{4\alpha(1-\lambda)3^k + [2\alpha(\lambda^2-1) - (\alpha-1)(1-\lambda)^2]2^{2k}}}.$$

In another case, if k = 1,2,3... and 0<α≤1, we have

$$\frac{2\alpha}{2^k(1-\lambda)} \le \frac{2\alpha}{\sqrt{4\alpha(1-\lambda)3^k + [2\alpha(\lambda^2-1) - (\alpha-1)(1-\lambda)^2]2^{2k}}}.$$

Citation: Shabani MM, Yazdi M, Sababe SH (2024) Coefficient estimates for a subclass of bi-univalent functions associated with the Salagean differential operator. Ann Math Phys 7(1): 091-095. DOI: https://dx.doi.org/10.17352/amp.000112

2. For the coefficient $|a_3|$, we make the following observations: If k=0 and 0<α≤1, we have

$$\frac{2\alpha^2}{2^{2k}(\lambda^2-1)+(2-2\lambda)3^k}\le\frac{4\alpha^2}{2^{2k}(1-\lambda)^2},$$

and

$$\frac{\alpha^2}{(1-\lambda)3^k}\le\frac{\alpha}{(1-\lambda)3^k}.$$

Then

$$\frac{2\alpha^2}{2^{2k}(\lambda^2-1)+(2-2\lambda)3^k}+\frac{\alpha^2}{(1-\lambda)3^k}\le\frac{4\alpha^2}{2^{2k}(1-\lambda)^2}+\frac{\alpha}{(1-\lambda)3^k}.$$

In another case, if k = 1,2,3... and 0<α≤1, we have

$$\frac{\alpha^2}{(1-\lambda)3^k}\le\frac{\alpha}{(1-\lambda)3^k}.$$

Then

$$\frac{4\alpha^2}{2^{2k}(1-\lambda)^2}+\frac{\alpha^2}{(1-\lambda)3^k}\le\frac{4\alpha^2}{2^{2k}(1-\lambda)^2}+\frac{\alpha}{(1-\lambda)3^k}.$$

Thus Theorem 3.2 clearly improves the estimate of coefficients $|a_2|$ and $|a_3|$ obtained by J.Jothibaso [6].

By setting $h(z)=p(z)=(\frac{1+z}{1-z})^{\alpha}$, k=λ=0 in Theorem 3.2. we get the following consequence.

Corollary 4.5: *Let the function f(z) given by the Taylor-Maclaurin series expansion (1.1) be in the bi-univalent function class* $\mathcal{S}_{\Sigma}^{*}(\alpha)$. *Then*

$$|a_2|\le\sqrt{2}\alpha,\ \text{and}\ |a_3|\le 3\alpha^2.$$

Remark 4.6: *Corollary 4.5 is an improvement of the following estimates obtained by the coefficient estimates for a well-known class* $\mathcal{S}_{\Sigma}^{*}(\alpha)$ *of strongly bi-starlike functions of order α as in [1].*

By setting $h(z)=p(z)=\frac{1+(1-2\beta)z}{1-z},(0\le\beta<1,z\in\mathbb{U})$ in Theorem 3.2. we get the following consequence.

Corollary 4.7: *Let the function f(z) given by the Taylor-Maclaurin series expansion (1.1) be in the bi-univalent function class* $\mathcal{S}_{\Sigma}^{h,p}(k,\lambda)$, *(0≤<1). Then*

$$|a_2|\le\min\left\{\frac{2(1-\beta)}{2^k(1-\lambda)},\ \sqrt{\frac{2(1-\beta)}{2^{2k}(\lambda^2-1)+(2-2\lambda)3^k}}\right\},$$

and

$$|a_3|\le\min\left\{\frac{4(1-\beta)^2}{2^{2k}(1-\lambda)^2}+\frac{1-\beta}{(1-\lambda)3^k},\frac{2(1-\beta)}{2^{2k}(\lambda^2-1)+(2-2\lambda)3^k}+\frac{1-\beta}{(1-\lambda)3^k}\right\}.$$

Remark 4.8: *Corollary 4.7 is an improvement of the following estimates obtained by J.Jothibaso [6].*

Corollary 4.9: *(see[6]) Let the function f(z) given by the Taylor-Maclaurin series expansion (1.1) be in the bi-univalent function class* $\mathcal{M}_{\Sigma}^{k,\lambda}(\beta)$. *Then*

$$|a_2|\le\sqrt{\frac{2(1-\beta)}{2^{2k}(\lambda^2-1)+(2-2\lambda)3^k}},$$

and

$$|a_3|\le\frac{4(1-\beta)^2}{2^{2k}(1-\lambda)^2}+\frac{1-\beta}{(1-\lambda)3^k}.$$

Corollary 4.10: *By setting k=λ=0 in Corollary 4.7, we have the coefficients estimates for the well-known class* $\mathcal{S}_{\Sigma}^{*}(\beta)$ *of bi-starlike functions of order β as in [1]. Further, taking k=1,λ=0 in Corollary 4.7, we obtain the estimates for the well-known class* $k_{\Sigma}(\beta)$ *of bi-convex functions of order β and our results reduce to [1].*

Conclusion

This paper introduces a new subclass of the function class Σ involving analytic and bi-univalent functions associated with the Salagean differential operator. Our study provides estimates for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions within this subclass, contributing to the advancement of knowledge in this area. The findings enhance recent research in the field and open up new avenues for further exploration and development in the theory of analytic and bi-univalent functions.

Acknowledgment

A part of this research was carried out while the Third author was visiting the University of Alberta.

References

1. Brannan DA, Taha TS. On some classes of bi-univalent functions. In: SM. Mazhar, A Hamoui, NS. Faour (Eds.), Mathematical Analysis and Its Applications, Kuwait; February 18-21, 1985, in KFAS Proceedings Series. Pergamon Press (Elsevier Science Limited), Oxford, 1988; 3:53-60.
2. Bulut S. Coefficient estimates for initial Taylor-Maclaurin coefficients for a subclass of analytic and bi-univalent functions defined by Al-Oboudi differential operator. ScientificWorldJournal. 2013 Dec 29;2013:171039. doi: 10.1155/2013/171039. PMID: 24487954; PMCID: PMC3893868.
3. Duren PL. Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.
4. Frasin BA. Coefficient bounds for certain classes of bi-univalent functions. Hacettepe Journal of Mathematics and Statistics. 2014; 43(3): 383-389.
5. Magesh N, Rosy T, Varma S. Coefficient estimate problem for a new subclass of biunivalent functions. J Complex Anal. 2013; 474231.
6. Jothibasu J. Certain Subclasses OF Bi-univalent Functions Defined By Salagean Operator. EJMAA. 2015; 3(1): 150-157.
7. Porwal S, Darus M. On a new subclass of bi-univalent functions. J Egyptian Math Soc. 2013; 21:190-193.
8. Orhan H, Magesh N, Balaji VK. Initial coefficient bounds for certain classes of meromorphic bi-univalent functions. Asian European J Math. 2014; 7(1):

Citation: Shabani MM, Yazdi M, Sababe SH (2024) Coefficient estimates for a subclass of bi-univalent functions associated with the Salagean differential operator. Ann Math Phys 7(1): 091-095. DOI: https://dx.doi.org/10.17352/amp.000112

9. Shabani MM, Hashemi Sababe S. On Some Classes of Spiral-like Functions Defined by the Salagean Operator. Korean J Math. 2020; 28: 137–147.

10. Shabani MM, Yazdi M, Hashemi Sababe S. Coefficient Bounds for a Subclass of Harmonic Mappings Convex in one direction. Kyungpook Math J. 2021; 61: 269-278.

11. Shabani MM, Yazdi M, Hashemi Sababe S. Some distortion theorems for new subclass of harmonic univalent functions. Honam Mathematical J. 2020; 42(4): 701–717.

12. Shabani MM, Hashemi Sababe S. Coefficient bounds for a subclass of bi-univalent functions associated with Dziok-Srivastava operator. Korean J Math. 2022; 30(1): 73–80.

13. Salagean GS. Subclasses of univalent functions, Lecture Notes in Math., Springer, Berlin. 1983; 1013: 362-372.

14. Srivastava HM, Murugusundaramoorthy G, Magesh N. On certain subclasses of biunivalent functions associated with hohlov operator. Global J Math Anal. 2013; 67–73.

Citation: Shabani MM, Yazdi M, Sababe SH (2024) Coefficient estimates for a subclass of bi-univalent functions associated with the Salagean differential operator. Ann Math Phys 7(1): 091-095. DOI: https://dx.doi.org/10.17352/amp.000112