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ANNALS OF Mathematics and Physics OPEN ACCESS JOURNAL

ISSN: 2689-7636 DOI: https://dx.doi.org/10.17352/amp

Review Article

On Λ-Fractional fluid mechanics

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Received: 12 January, 2024
Accepted: 25 April, 2024
Published: 26 April, 2024

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Keywords: Λ-fractional derivative; Λ-fractional space; Λ-fractional derivative; Fractional field theorems; Λ- fractional continuum mechanics; Λ-fractional fluid mechanics; Λ-fractional newtonian fluids; Λ-fractional Navier-Stokes equations; Λ-Fractional Darcy flow

Abstract

Λ-fractional analysis has already been presented as the only fractional analysis conforming with the Differential Topology prerequisites. That is, the Leibniz rule and chain rule do not apply to other fractional derivatives; This, according to Differential Topology, makes the definition of a differential impossible for these derivatives. Therefore, this leaves Λ-fractional analysis the only analysis generating differential geometry necessary to establish the governing laws in physics and mechanics. Hence, it is most necessary to use Λ-fractional derivative and Λ-fractional transformation to describe fractional mathematical models. Other fractional "derivatives" are not proper derivatives, according to Differential Topology; they are just operators. This fact makes their application to mathematical problems questionable while Λ-derivative faces no such problems. Basic Fluid Mechanics equations are studied and revised under the prism of Λ-Fractional Continuum Mechanics (Λ-FCM). Extending the already presented principles of Continuum Mechanics in the area of solids into the area of fluids, the basic Λ-fractional fluid equations concerning the Navier-Stokes, Euler, and Bernoulli flows are derived, and the Λ-fractional Darcy's flow in porous media is studied. Since global minimization of the various fields is accepted only in the Λ-fractional analysis, shocks in the Λ-fractional motion of fluids are exhibited.

Introduction

Fractional derivatives and integrals [1-6] have been applied in many fields since they are considered more advanced mathematical tools for formulating realistic responses to various scientific problems in Physics and Engineering [7-9]. Especially in mechanics, researchers work in disordered (non-homogeneous) materials with microstructure, Vardoulakis, et al. [10], Wyss, et al. [11], have used fractional analysis for a better description of the mechanics of porous materials, colloidal aggregates, ceramics, etc., since major factors in determining materials' deformation are microcracks, voids, and material phases. Further viscoelasticity problems have been recently formulated by applying Fractional Analysis.

Generally, those problems demand nonlocal theories. Just to satisfy that requirement, gradient strain theories appeared, Toupin [12], Mindlin [13], Aifantis [14], and Eringen [15]. In these theories, the authors introduced intrinsic material lengths that accompany the higher-order derivatives of the strain. Many problems have been solved employing those theories concerning size effects, lifting various singularities, porous materials, Aifantis [14,16,17], Askes & Aifantis [18], and Lazopoulos [19,20]. Another nonlocal approach was introduced by Kunin [21,22].

There are many studies considering fractional elasticity theory, introducing fractional strain, Drapaca, et al [23], Carpinteri, et al. [24,25], Di Paola, et al. [26], Atanackovic, et al. [27], Agrawal [28], Sumelka [29]. Baleanu and his co-workers [30-32] have presented a long list of publications concerning various applications of fractional calculus in physics and control

Citation: Lazopoulos A, Lazopoulos K (2024) On Λ-Fractional fluid mechanics. Ann Math Phys 7(1): 107-117. DOI: https://dx.doi.org/10.17352/amp.000114

theory to solve differential equations and numerical solutions. In addition, Tarasov [33,34] has presented a Fractional Vector Field theory with applications.

Lazopoulos [35] introduced Fractional derivatives of the strain in the strain energy density function in an attempt to introduce non-locality in the elastic response of materials. Fractional calculus was used by many researchers, not only in the field of Mechanics but mainly in Physics and especially in Quantum Mechanics, to develop the idea of introducing non-locality. The history of fractional calculus is dated to the 17th century. Particle physics, electromagnetics, mechanics of materials, Hydrodynamics, fluid flow, rheology, viscoelasticity, optics, electrochemistry and corrosion, and chemical physics are some fields where fractional calculus has been introduced.

Nevertheless, the formulation of the various physical problems into the context of Fractional Mathematical Analysis follows a procedure that might be questionable. Although the various laws in Physics have been derived through differentials, this is not the case for the well-known fractional derivatives which are not related to differentials. Simple substitutions of the conventional differentials to fractional ones cannot express the realistic behavior of the various physical problems. Lately, Lazopoulos [36] proposed the fractional Λ-derivative, a modification of the fractional L-derivative, along with the conjugate fractional Λ-space, where the fractional Λ-derivative behaves with conventional derivative rules.

Specifically, derivatives in Fractional Calculus are merely operators and not derivatives since none satisfies the criteria of the differential topology of a derivative, as described in Chillingworth [21]. Those criteria are:

1. Linearity $D\left(af\left(x\right)+bg\left(x\right)\right)=aDf(x)+bDg(x)$ (1)

2. Leibniz rule $D\left(f\left(x\right)\cdot g\left(x\right)\right)=Df(x)\cdot g\left(x\right)+f(x)\cdot Dg(x)$ (2)

3. Chain rule $D\left(g\left(f\right)\right)(x)=Dg(f(x)\cdot Df(x)$ (3)

This great weakness was deplored single-mindedly in the past by various specialists in the field (Samko, et al. [22], König, et al. ([23,24]), Cresson, et al. [25]). Serious efforts to tackle this grave problem by designing fractional derivatives that follow Leibniz's rule (Jumarie [26,27], Yang [28]) were made without success. Therefore, as mentioned above, the problem is severe since it forbids the definition of a differential and, consequently, differential geometry.

It is repeatedly pointed out in every differential topology book (e.g., Gauld, D. [29] p.95) that the Leibniz rule and the chain rule are essential to define a differential, tangent space, and, consequently, geometry. On the other hand, the Λ-fractional derivative is the only proper fractional derivative since it follows the differential topology perquisites for defining a differential in Λ-space.

The theory of Λ-fractional elastic solid mechanics has been presented by Lazopoulos [37]. Further, globally stable states have been defined by Lazopoulos [38]. Those states have been defined in the expansion of spherical balloons, Lazopoulos [39].

The present work deals with the study of fractional fluid flow, introducing the Λ-fractional derivative into the various flows concerning the fractional Navier-Stokes equations and the fractional Euler and Bernoulli ones. Furthermore, the fractional flows are introduced to Darcy's flows, just to study the flows through porous media. A long list of works concerning fractional fluid dynamics exists [39,40,41]. However, using fractional derivatives in conventional fluid flow laws exhibits the handicap of having no differentials. Therefore, the novelty of this work is to apply a proper, according to Differential Topology, fractional derivative (Λ-fractional derivative) to the field of hydrodynamics. This would not only define a differential but also legitimize the use of this fractional derivative. After a brief introduction of the Λ-Fractional derivative and Fractional Vector field theory, the fractional fluid flows are studied using the Λ-fractional analysis, trying to introduce nonlocal fluid mechanics both in time and space considering inhomogeneities, porous, cracks, etc.

The Λ-fractional derivative

This chapter presents a brief outline of fractional calculus, while the interested reader refers to refs [6-10], for further information. The left and right fractional integrals are defined by,

$$ {}_{a}I_{x}^{\gamma}f(x)=\frac{1}{\Gamma(\gamma)}\int_{a}^{x}\frac{f(s)}{(x-s)^{1-\gamma}}ds, \qquad (1)4$$

$$ {}_{x}I_{b}^{\gamma}f(x)=\frac{1}{\Gamma(\gamma)}\int_{x}^{b}\frac{f(s)}{(s-x)^{1-\gamma}}ds, \qquad (2)5$$

γ is the order of fractional integrals with where $\Gamma(x)=(x-1)!$ with $\Gamma(\gamma)$ Euler's Gamma function. In addition, the left and right Riemann-Liouville (RL) derivatives are defined by:

$$ {}_{a}^{RL}D_{x}^{\gamma}f(x)=\left(\frac{d}{dx}\right)^{m+1}\left({}_{a}I_{x}^{m-\gamma}\ f(x)\right), \qquad (3)6$$

and

$$ {}_{x}^{RL}D_{b}^{\gamma}f(x)=\left(-\frac{d}{dx}\right)^{m+1}\left({}_{x}I_{b}^{m-\gamma}\ f(x)\right). \qquad (4)7$$

Let us point out that for the left fractional integrals and derivatives

$$ {}_{a}^{RL}D_{x}^{\gamma}\left({}_{a}\mathrm{I}_{x}^{\gamma}f(x)\right)=f(x). \qquad (5)8$$

A similar relation is valid for the right RL-fractional derivative and right fractional integral. Considering only the left space, the Λ-fractional derivative (Λ-FD) has been defined as

$$ {}_{a}^{\Lambda}D_{x}^{\gamma}f(x)=\frac{{}_{\alpha}^{RL}D_{x}^{\gamma}f(x)}{{}_{a}^{RL}D_{x}^{\gamma}x^{\gamma}}. \qquad (6)9$$

Recalling the definition of the Riemann-Liouville fractional derivative, Eq. (3), the Λ-FD is expressed by,

Citation: Lazopoulos A, Lazopoulos K (2024) On Λ-Fractional fluid mechanics. Ann Math Phys 7(1): 107-117. DOI: https://dx.doi.org/10.17352/amp.000114

$$ {}_a^{\Lambda}D_x^{\gamma} f\left(x\right) = \frac{\dfrac{d\, {}_aI_x^{1-\gamma} f(x)}{dx}}{\dfrac{d\, {}_aI_x^{1-\gamma} x}{dx}} = \frac{d\, {}_aI_x^{1-\gamma} f(x)}{d\, {}_aI_x^{1-\gamma} x}. \quad (7)10 $$

Further, if

$$ X = {}_aI_x^{1-\gamma} x \text{ and } F(X) = {}_aI_x^{1-\gamma} f(x), \quad (8)11 $$

the Λ-FD behaves as a conventional derivative in the fractional Λ-space *(X, F(X))* with local properties. Fractional Differential Geometry may be developed as a conventional differential geometry in the Λ-fractional space (*X, F(X)*). Therefore it is a proper derivative in Λ-space. Hence we can transfer a mathematical model to Λ-space, solve the problem there, and transfer the results back to the initial space.

Indeed, Eq. (8a) yields,

$$ X = \frac{x^{2-\gamma}}{\left(2-3\gamma+\gamma^2\right)\Gamma(1-\gamma)}, \quad (9)12 $$

In addition, Eqs.(8b,9) suggest that:

$$ F\left(x\right) = {}_{\alpha}I_x^{1-\gamma} f\left(x\right) = \frac{1}{\Gamma(1-\gamma)} \int_{\alpha}^{x} \frac{f(s)}{(x-s)^{\gamma}} ds, \quad (10)13 $$

Inverting Eq.(9) it appears,

$$ x = (\left(2-3\gamma+\gamma^2\right)\Gamma\left(1-\gamma\right)X)^{1/(2-\gamma)} = x(X), \quad (11)14 $$

Proceeding further to the definition of the fractional Λ-space, inserting *x(X)* into Eq.(10), the function *F(x)* may be expressed as a function of *X*.

$$ F\left(X\right) = F\left(x\left(X\right)\right), \quad (12) $$

It is evident that, in the just presented Λ-fractional derivatives, only left fractional integrals and RL fractional derivatives were considered. If we were to involve the right fractional integrals and RL derivatives, then the Λ-Fractional derivatives should be defined by

$$ {}_{\alpha}^{\Lambda}D_{\chi}^{\gamma} f\left(x\right) = \frac{d\, {}_aI_x^{1-\gamma} f(x)}{2d\, {}_aI_x^{1-\gamma} x} = \frac{dF(X)}{dX}. \quad (13a) $$

with

$$ F\left(x\right) = \frac{{}_{\alpha}I_x^{1-\gamma} f\left(x\right)}{2} = \frac{1}{\Gamma\left(1-\gamma\right)} \left(\int_{\alpha}^{x} \frac{f\left(s\right)}{\left(x-s\right)^{\gamma}} ds \right) = F\left(x\left(X\right)\right). \quad (13b) $$

It will be clarified in the application how from the initial space *(x, f(x))* the fractional Λ-space *(X, F(X))* is defined. Furthermore, the pullback of the results in the initial space will also be demonstrated. For simplicity reasons, only the left fractional integrals and derivatives will be taken into consideration. Nevertheless, applications with symmetric space may be found in [21].

Geometry in the Λ-fractional space

Just to understand what happens in the Λ-fractional space, the geometry of the surface,

$z = x^2y^2,\ 0<x<1,\ 0<y<1,$ (14)

will be discussed.

The fractional Λ-space (*X, Y, Z*) is defined by,

$$ X = \frac{x^{2-\gamma}}{\left(2-3\gamma+\gamma^2\right)\Gamma(1-\gamma)} \quad (15) $$

$$ Y = \frac{y^{2-\gamma}}{\left(2-3\gamma+\gamma^2\right)\Gamma(1-\gamma)} \quad (16) $$

$$ Z = {}_bI_y^{1-\gamma}\, {}_{\alpha}I_x^{1-\gamma} z\left(x,y\right) = \frac{1}{\left(\Gamma\left(1-\gamma\right)\right)^2} \int_b^y \left(\int_{\alpha}^{x} \frac{z(s,t)}{(x-s)^{\gamma}} ds \right) \frac{dt}{(y-t)^{\gamma}}, \quad (17) $$

ith *a=b=0* , Eq.(17) yields,

$$ Z = \left(-\frac{2\left(\left(X\Gamma(3-\gamma)\right)^{\frac{1}{2-\gamma}}\right)^{3-\gamma}}{\Gamma(4-\gamma)} Y\right). \quad (18) $$

For γ=0.6, the surface Z in the Λ-fractional space is defined by

$Z = 0.947X^{1.714}Y^{1.714}$ (Figure 1) (19)

and it is shown in Figure 2.

Further, the tangent space of the surface with γ=0.6, at the point *X=Y=0.6* is defined by,

$$ Z = (0.947X^{1.714}Y^{1.714})_{(X=Y=0.6)} + \frac{dZ\left(X=Y=0.6\right)}{dX}\left(X-0.6\right) + \frac{dZ\left(X=Y=0.6\right)}{dY}\left(Y-0.6\right) \quad (20) $$

and finally, the equation of the tangent space in the Λ-fractional space,

$Z = 0.164 + 0.469(X-0.6) + 0.469(Y-0.6)$ (Figure 3). (21)

The corresponding surface in the initial space to the tangent plane in the Λ-fractional space is defined by,

Citation: Lazopoulos A, Lazopoulos K (2024) On Λ-Fractional fluid mechanics. Ann Math Phys 7(1): 107-117. DOI: https://dx.doi.org/10.17352/amp.000114

$$z = x^2 y^2{}_{(x=y=0.81)} + \left({}^{RL}_{\ 0}D^{1-\gamma}_{y=0.81}\ {}^{RL}_{\ 0}D^{1-\gamma}_{x=0.81} \left(\frac{dZ}{dX} \right) \right) \left(X(x) - 0.6 \right) + \left({}^{RL}_{\ 0}D^{1-\gamma}_{y=0.81}\ {}^{RL}_{\ 0}D^{1-\gamma}_{x=0.81} \left(\frac{dZ}{dY} \right) \right) \left(Y(y) - 0.6 \right). \qquad (22)$$

The surface defined by Eq.(22) is shown in Figure 4

It seems that the initial surface and the tangent surface corresponding to the tangent space at the Λ-space have almost a common tangent plane in the initial space since their mathematical expressions are different but quite close.

The fractional field theorems

The conventional field theorems are expressed by:

a. Green's theorem: Let $Q_x(x,y)$, $Q_y(x,y)$, be smooth real functions in a domain Ω, with its boundary a smooth closed curve $\partial\Omega$. Then,

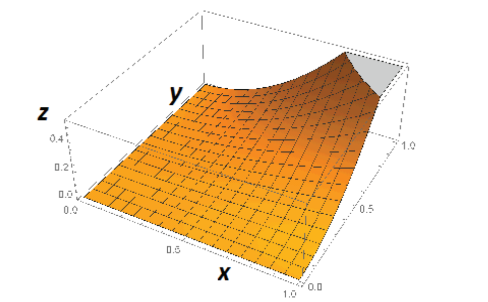

Figure 1: The surface z.

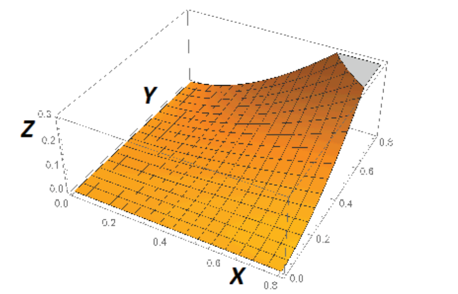

Figure 2: The surface Z in the Λ- fractional space.

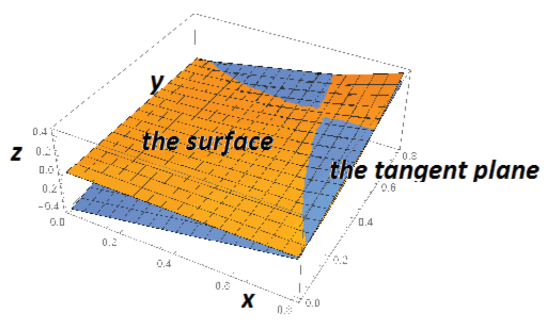

Figure 3: The surface with the tangent space in the Λ-fractional space.

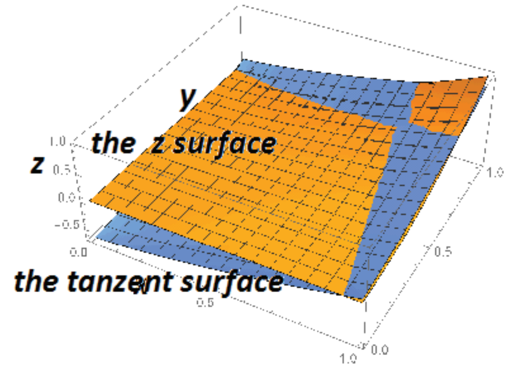

Figure 4: The surface with its tangent surface at the point (x=y=0.8106) at the initial space.

$$\int_{\partial\Omega} \left(Q_x dx + Q_y dy \right) = \iint_{\Omega} dxdy \left(\frac{dQ_x}{dy} - \frac{dQ_y}{dx} \right). \qquad (23)$$

+Corollary: When $Q_x(x,y)$, $Q_y(x,y)$, are derived by a potential function Φ(x,y) with $Q_x = \frac{d\Phi}{dx}, Q_y = \frac{d\Phi}{dy}$, the RHS of Eq.(23) becomes zero. That means that the curvilinear integral along a closed smooth boundary is zero.

b. Stoke's theorem: For a smooth vector fiel $\partial\Omega$' defined on a simple surface Ω with the boundary , Stoke's theorem is expressed by,

$$\int_{\partial\Omega} \left(\boldsymbol{F}, d\boldsymbol{L} \right) = \iint_{\Omega} (\nabla \times \boldsymbol{F}, d\boldsymbol{S}). \quad (24)$$

where, $(\cdot,\cdot)$ denotes the scalar product.

c. The Gauss' (divergence) theorem: For a space region Ω with smooth surface boundary $\partial\Omega$, the volume integral of the divergence of a vector field $\boldsymbol{F}$ over Ω is equal to the surface integral of $\boldsymbol{F}$ over the boundary $\partial\Omega$:

$$\int_{\partial\Omega} \left(\boldsymbol{F}, d\boldsymbol{S} \right) = \iiint_{\Omega} \nabla \cdot \boldsymbol{F} d\Omega \quad (25)$$

Although the field theorems are valid in the fractional Λ-space, they are not necessarily valid in the initial space. Nevertheless, the Λ- space results may be pulled back into the initial space.

The balance principles

Before discussing the general problem of hydrodynamics with its specific principles, it is pointed out that geometry and, consequently, mechanics have meaning in theΛ-fractional space. The results of the various analyses in the Λ-fractional space are transferred as functions in the initial space. No derivatives of the Λ-space have any meaning in the initial space as derivatives. However, they may have a meaning as functions.

Almost all balance principles are based on Reynold's transport theorem. Hence the modification of that theorem,

Citation: Lazopoulos A, Lazopoulos K (2024) On Λ-Fractional fluid mechanics. Ann Math Phys 7(1): 107-117. DOI: https://dx.doi.org/10.17352/amp.000114

just to conform to fractional analysis is presented. The conventional Reynold's transport theorem is expressed by:

$$\frac{D}{DT}\int_W A dV = \int_W \frac{DA}{DT} dV + \int_{\partial W} A\mathbf{V}_n dS. \tag{26}$$

For a vector field A applied upon region W with boundary ∂W and $\mathbf{V}_n$ is the normal velocity of the boundary ∂W.

a. **The balance of mass:** The conventional balance of mass, expressing the mass preservation is expressed by:

$$\frac{D}{DT}\int_W R dV = 0 \tag{27}$$

Recalling the fractional Reynold's Transport Theorem, we get:

$$\frac{D}{DT}\int_W R dV = \int_W (\partial_T R(\mathbf{X},T) + Div[\mathbf{V}R]) dV \tag{28}$$

Since Eq. (28) is valid for any volume V, the continuity equation is:

$$\partial_T R + Div[\mathbf{V}R] = 0. \tag{29}$$

where Div is defined in the Λ-fractional space. That is the continuity equation expressed in fractional form.

b. **Balance of linear momentum principle:** It is reminded that the conventional balance of linear momentum is expressed in continuum mechanics by:

$$\frac{D}{DT}\int_W R\mathbf{V}\delta V = \int_{\partial\Omega} \mathbf{T} dS + \int_\Omega R\mathbf{B} dV \tag{30}$$

where $\mathbf{V}$ is the velocity, $\mathbf{T}$ is the traction on the boundary and $\mathbf{B}$ is the body force per unit mass. Likewise, that principle in fractional form is expressed by:

$$\frac{D}{DT}\int_\Omega R\mathbf{V} dV = \int_\Omega \left[R\mathbf{B} + Div(\mathbf{T})\right] dV \tag{31}$$

Hence the equation of linear motion, expressing the balance of linear momentum is defined by,

$$Div\left[\mathbf{T}\right] + R\mathbf{B} - R\frac{D\mathbf{V}}{DT} = 0 \tag{32}$$

In Eq.(32) the term $\mathbf{T}$ is introduced by the pressure P. That is affected by adding the term $-P\mathbf{I}$. Therefore Eq.(32) becomes:

$$Div\left[\mathbf{T} - P\mathbf{I}\right] + R\mathbf{B} - R\frac{d\mathbf{V}}{dT} = 0 \tag{33}$$

Following similar steps as in the conventional case, the balance of rotational momentum yields the symmetry of the Cauchy stress tensor.

First law of thermodynamics

This law occurs from Eq.(42). Firstly we replace $\mathbf{T} - p\mathbf{I}$ with,

$$\Sigma = \mathbf{T} - p\mathbf{I}, \Sigma = \left[\Sigma_{ij}\right] \tag{34}$$

Then the equation of conservation of energy, taking into consideration the principle of conservation of linear momentum, Eq.(33), yields the rate of change of internal energy in the Λ-fractional space as the sum of stress power plus the heat added by the continuum. The vector **C** is defined in the Λ-fractional space is defined as the heat flux per unit area per unit time by conduction and Z is per the radiant heat constant unit mass per unit time. Further, The caloric equation of state is expressed by $e = e(R,T)$. Finally, the equation of the conservation of energy is defined by, see Ref. [46].

$$\frac{D}{DT}\left(\frac{\mathbf{V}^2}{2} + e\right) = \frac{1}{R} D_j\left(\Sigma_{ij}V_i\right) + B_i V_i - \frac{1}{R} D_i C_i + Z. \tag{35}$$

The various results in the Λ-fractional space should be transferred as functions in the initial space.

Navier-stokes equations

The Navier-Stokes Equations consist of the following equations:

a) Balance of mass (continuity) Eq.(29).

b) Balance of linear momentum, Eq.(33)

c) The first law of thermodynamics, Eq (35)

d) Constitutive equations

$$\begin{aligned} T_{xx} &= \lambda\nabla\ \mathbf{V} + 2\mu D_x U \\ T_{yy} &= \lambda\nabla\mathbf{V} + 2\mu D_y V \\ T_{zz} &= \lambda\nabla\mathbf{V} + 2\mu D_z W \\ T_{xy} &= T_{yx} = \mu\left[D_x V + D_y U\right] \\ T_{xz} &= T_{zx} = \mu\left[D_z U + D_x W\right] \\ T_{yz} &= T_{zy} = \mu\left[D_y W + D_z V\right] \end{aligned} \tag{36}$$

e) The kinetic equation of state:

$$p = p\left(\rho, \mathrm{T}\right) \tag{37}$$

f) The Fourier law of heat conduction:

$$\mathbf{c} = -k\nabla^{(\alpha)}T \tag{38}$$

g) The caloric equation of state:

$$e = e\left(R, \mathrm{T}\right) \tag{39}$$

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The system of these sixteen equations contains 16 unknowns therefore it is determinate. Usually, the studies concerning the Newtonian fluid, are restricted to the equations of conservation of mass (29), linear momentum (33), and constitutive equations (36). The solution in the Λ-fractional space is transferred to the initial space.

The Euler and Bernoulli equations

The Euler equations occur from Eq.(30) for non-viscous fluid, for which **T**=0. This holds because the Lamé constants λ, and μ are considered zero in this case. Therefore, Eq.(33) takes the form:

$$R\frac{D}{DT}\mathbf{V} = R\mathbf{B} - Div\left[P\mathbf{I}\right]. \tag{40}$$

Let's assume a barotropic condition $R=R(P)$. A pressure function may be defined as:

$$P\left(R\right) = \int_{P_0}^{P}\frac{DP}{R} \tag{41}$$

Moreover, the body force may be described by a potential function:

$$\mathbf{B}=-\nabla\Omega, \tag{42}$$

with these two conditions the conservation of linear momentum, Eq. (33), becomes:

$$\frac{D}{DT}\mathbf{V} = -\nabla\left(\Omega + R\right). \tag{43}$$

If Eq.(43) is integrated along a streamline, the result is the Bernoulli equation in the fractional form:

$$\Omega + R + \frac{U^2}{2} + \int\frac{D}{DT}V_i DX_i = C\left(T\right). \tag{44}$$

Any problem may be solved following the conventional way in the Λ-fractional space, however, the results should be transferred into the initial space.

Darcy's law

Consider a viscous fluid, flowing in a straight pipe of a constant circular cross-section of inner radius R and cross-section of $A = \pi R^2$ with perimeter $\Gamma = 2\pi R$. If L is the length of the considered segment of the pipe defined by positions x=0 and x=a, then $l=b-\alpha$. Transferring the length into the Λ-fractional space,

$$L = I_l^{1-\gamma}x = \frac{l^{2-\gamma}}{\Gamma(3-\gamma)} \tag{45}$$

Denoting $Q(X, T)$ the discharge of the fluid in the Λ-space, continuity of flow demands that $Q(X, T)$ be the same at any cross-section of the pipe segment in the Λ-space, depending only upon time

$$Q(X,T)=Q(T) \tag{46}$$

The cross-sectional velocity of the flow in the Λ-space may be computed by:

$$V = \frac{Q}{A} = \frac{DU(T)}{DT}. \tag{47}$$

Due to the fluid's internal friction, shear stresses τ are developed, between the fluid and the inner pipe-wall. Similarly to fractional viscoelasticity, the interface friction shear stress in circular pipes is equal to:

$$\tau = \frac{4\mu_f}{R}V = \frac{4\mu_f}{R}\frac{DU(T)}{DT} \tag{48}$$

Where μ_f is fluid viscosity. The friction stresses may be substituted by a fictitious body force per unit length of the pipe in the Λ-fractional space,

$$F_b = \tau \cdot V = 8\pi\mu_f\frac{DU(T)}{DT} \tag{49}$$

The conservation of linear momentum equation in the Λ-fractional space yields:

$$-\frac{\partial P}{\partial X}ADX - F_b DX = R_f AVDX = R_f A\frac{D^2U(T)}{DT^2}DX, \tag{50}$$

or

$$F = CV = C\frac{DU(T)}{DT}. \tag{51}$$

Since, according to Poiseille's law, F is proportional to the mean flow velocity

$$F = CV = C\frac{DU(T)}{DT}. \tag{52}$$

Furthermore, the coefficient c of viscous friction is proportional to the fluid viscosity and inversely proportional to the square of the pipe radius:

$$C = \pi\cdot\frac{\mu_f}{R^2}. \tag{53}$$

Thus we obtain the differential equation:

$$-\frac{\partial P}{\partial X} = R_f\frac{D^2U(T)}{DT^2} + C\frac{DU(T)}{DT}. \tag{54}$$

Considering:

$$J(t) = R_f\frac{D^2U(T)}{DT^2} + C\frac{DU(T)}{DT} \tag{55}$$

The governing Eq.(54) becomes:

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$$-\frac{\partial P}{\partial X} = J(t) \quad (56)$$

For the case that $J(t)=J$ (constant) between the segments x=*o* and x=*l*, the fluid velocity is defined by:

$$\frac{D^2U(T)}{DT^2} + C\frac{DU(T)}{DT} = \dot{J} = constant. \quad (57)$$

Solution of Fractional D.E:

In case Δp is the pressure drop from one end to the other of the pipe segment with:

$$\Delta P = P(A) - P(0) < 0 \quad (58a)$$

Then the total discharge is defined by:

$$Q = AV = -(\frac{A}{c})\frac{\Delta P}{L} \quad (58b)$$

Eq.(58b) is the famous Darcy's law in the Λ-fractional space. Transferring the flow rate from the Λ-fractional space to the initial space, considering the action of two fractional variables, with fractional order γ_1 of time *t* and γ_2 of space *x*, the flow rate along the pipe is defined by,

$$q(x,t) = -(\frac{A}{c})\frac{\Delta P}{L}x^{1-\gamma_2}t^{1-\gamma_1}. \quad (58c)$$

Fractional flow in porous media

Considering the difference ΔP to the hydrostatic pressure we get:

$$\Delta P = -R_f g\Delta H. \quad (59)$$

Then, we may assume that for the wetted area A_v (The area of the voids), Darcy's law may be applied with:

$$Q = -(\frac{A_v}{c})\frac{\Delta P}{L}. \quad (60)$$

Assuming that the surface porosity Φ_A is equal to volume porosity Φ we get:

$$\Phi_A = \frac{A_v}{A} \approx \frac{V_v}{V} = \Phi. \quad (61)$$

And thus:

$$A_v \approx \Phi \cdot A \quad (62)$$

For the specific fluid discharge:

$$Q_A = \frac{Q}{A}. \quad (63)$$

we get:

$$V = \frac{Q}{A_v} \approx \frac{Q}{A}\frac{A}{A_v} = \frac{Q_A}{\Phi} \rightarrow Q_A = \Phi V. \quad (64)$$

Hence Darcy's law becomes:

$$Q_A = \frac{Q}{A} = -K\frac{\Delta P}{L}, \quad (65)$$

With $K = \frac{\Phi}{c}$. Finally, Darcy's law may be generalized by:

$$Q_A = -\frac{1}{f}\frac{\partial P}{\partial X}, \; where\; f = \frac{\mu_f}{K} \quad (66)$$

with *P*: The pore fluid pressure.

Q_A: The specific fluid discharge.

K: The permeability of the porous medium.

μ_f: The viscosity of the fluid.

Transferring the result into the initial space with fractional order γ_1 of time *t* and γ_2 of space *x*, the specific flow is defined by,

$$q(x,t) = \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\frac{d}{dt}\int_0^t \frac{1}{(t-\tau)^{1-\gamma_1}}(\frac{d}{dx}\int_0^x \frac{Q_A(s,\tau)}{(x-s)^{1-\gamma_2}}ds)dt \quad (67)$$

Fractional flow in elastic tubes

Consider a thin elastic tube, at its (unstressed) reference placement in the Λ-fractional space, with its inner radius *R* and its thickness Δ<<*R*. The elastic tube is considered in the context of linear elasticity with modulus *E*. Loading the tube with pressure *P*, its radius increases by Δ*R*. Hence its cross-sectional area becomes:

$$A = \pi \cdot (R + \Delta R)^2, \quad (68)$$

and its current radius $\tilde{R} = R + \Delta R$ is given by:

$$\tilde{R} = R + \Delta R = R + \frac{R^2}{E \cdot \Delta}p \quad (69)$$

Restricting to the linear elasticity where the changes of the radius are infinitesimal, we get:

$$A = A_0 + \Delta A = A_0(1 + 2\frac{R_0 \cdot P}{E \cdot \Delta}). \quad (70)$$

Therefore the relation between the variable cross-section area and the fluid pressure and cross-section area:

$$A = A_0 \cdot \left(1 + \frac{P}{K}\right), \quad K = \frac{2 \cdot R}{E \cdot \Delta}. \quad (71)$$

Since mass balance is expressed by:

Citation: Lazopoulos A, Lazopoulos K (2024) On Λ-Fractional fluid mechanics. Ann Math Phys 7(1): 107-117. DOI: https://dx.doi.org/10.17352/amp.000114

$$\frac{\partial A}{\partial T}+\frac{\partial Q}{\partial X}=0. \qquad (72)$$

Recalling Darcy's law with:

$$Q=-\left(\frac{A}{C}\right)\frac{\partial P}{\partial X},\ c=\pi^2\frac{\mu_f}{A}, \qquad (73)$$

or

$$Q=-\frac{A^2}{\pi^2\mu_f}\frac{\partial P}{\partial X}. \qquad (74)$$

Eliminating the discharge Q, we end up to:

$$\frac{\partial A}{\partial T}=\frac{1}{\pi^2\mu_f}\frac{\partial(A^2\frac{\partial P}{\partial X})}{\partial X}. \qquad (75)$$

Since:

$$\frac{\partial A}{\partial T}=\frac{\partial A}{\partial P}\frac{\partial P}{\partial T}=\frac{A_0}{\tilde{K}}\frac{\partial P}{\partial X}. \qquad (76)$$

Consequently, Eq. (75) yields:

$$\frac{\partial P}{\partial T}=\frac{1}{\pi^2\mu_f}(2\cdot K\cdot\left(\frac{\partial P}{\partial X}\right)^2+A\cdot\tilde{K}\cdot\frac{\partial^2 P}{\partial X^2}). \qquad (77)$$

However, for small changes in pressure, the linear problem is considered, resulting in: $\frac{\partial P}{\partial T}=c_p\frac{\partial^2 P}{\partial X^2}$, (78)

where:

$$c_p=\frac{A_0\cdot\tilde{K}}{\pi^2\mu_f}=\frac{2\cdot R^3}{\pi\cdot\mu_f\cdot E\cdot\Delta}. \qquad (79)$$

Solution of parabolic equation

Following just the same procedure, but for the fractional time derivative fields, we get the fractional parabolic equation,

$$\frac{dP}{dT}=c_p\frac{\partial^2 P}{\partial X^2}. \qquad (80)$$

Furthermore, the initial condition expresses the constant pressure value,

$$T=0:\ P=P_0\ ,\ \forall X\ \varepsilon\left[0,L\right]. \qquad (81)$$

Further, at the time = 0⁺, the pressure at the entry point of the tube is increased by ΔP, keeping constant the pressure value at the exit point of the tube. Consequently, the boundary conditions for $T>o$ become,

$$X=0, P=P_1=P_o+\Delta P\ , \qquad (82)$$

$$X=L, P=P_0. \qquad (83)$$

Nondimensionaling the variables we get:

$$P^*=\frac{P}{P_o},\ X^*=\frac{X}{L}\ \left(0\le X^*\le 1\right),\ and\ T^*=\frac{T}{T_c}\text{ where }T_c=\frac{L^2}{c_p}. \qquad (84)$$

Omitting the upper stars the nondimensional governing equation of the fractional flow becomes,

$$\frac{dP}{dT}=\frac{\partial^2 P}{\partial X^2}. \qquad (85)$$

with the initial condition,

$$X=0\ ;\ P=1\ ,\forall X\ \varepsilon\left[0,1\right] \qquad (86)$$

and the boundary conditions,

$$X=0\ ,P=P_1=1+\lambda\ ,\ \lambda=\frac{\Delta P}{P_0}, \qquad (87)$$

$$X=1,\ P=P_2=1.$$

The initial condition for the pressure,

$$T=0:\ P=P_o\ ,\ \forall X\ \varepsilon\left[0,L\right]. \qquad (88)$$

At T = 0⁺, the pressure at the entry point is increased by Δp with constant pressure at the exit point. Hence the boundary conditions for T>0 become:

$$X=0, P=P_1\pm P_o+\Delta P \qquad (89)$$

$$X=L,\ P=P_0$$

Non-dimensional variables become:

$$P^*=\frac{P}{P_o},\ X^*=\frac{X}{L}\ \left(0\le X^*\le 1\right),\ and\ T^*=\frac{T}{T_c}\text{ where }T_c=\frac{L^2}{c_p}. \qquad (90)$$

The non-dimensional equation becomes:

$$\frac{\partial P}{\partial T}=\frac{\partial^2 P}{\partial X^2}. \qquad (91)$$

With the initial conditions.

$$T=0:\ P=1\ ,\ \forall X\ \varepsilon\left[0,L\right] \qquad (92)$$

And boundary conditions:

$$X=0,\ P=P_1\pm 1+\lambda\ ,\ \lambda=\frac{\Delta P}{P_0}\ , \qquad (93)$$

$$X=1,\ P=P=1.$$

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Considering first the steady solution:

$$\frac{DP}{DT}=0\quad \frac{D^2P}{DX^2}=0\; P=\overline{P}=P_1+(P_1-P_2)X, \tag{94}$$

and introducing a renormalized pressure:

$$\hat{P}=\frac{P-\overline{P}}{P_1-P_2}, \tag{95}$$

from the above formulas occur:

$$\frac{\partial\hat{P}}{\partial T}=\frac{\partial^2\hat{P}}{\partial X^2},\quad 0\le X\le 1. \tag{96}$$

The initial condition is:

$$\hat{P}=H\left(X\right)=X-1,\; T=0,\; 0\le X\le 1, \tag{97}$$

and the boundary conditions:

$$\hat{P}=0, T>0, X=0\; and\; X=1\,. \tag{98}$$

Applying the separation of variables technique for the fractional diffusion equation we get:

$$\hat{P}(X,T)=\hat{X}\left(X\right)\hat{T}(T). \tag{99}$$

Following the conventional procedure for the diffusion equation, we get:

$$\frac{1}{\hat{X}}\frac{D\hat{X}}{DX}=\frac{1}{\hat{T}}\frac{D^2\hat{T}}{DT^2}=-\lambda^2. \tag{100}$$

Due to the appendix, the solution to the Eq. (100) is given by:

$$\hat{P}\left(X,T\right)=\Sigma_{n=1}^{\infty}b_n Sin\left(n\pi X\right)\Phi^{\alpha}(-\left(n\pi\right)^2,T). \tag{101}$$

Where:

$$\Phi^{\alpha}\left(-\left(n\pi\right)^2,T\right)=\Sigma_{k=1}^{\infty}(-\left(n\pi\right)^2)^{\kappa}\frac{\Gamma\left(\kappa+1-\alpha\right)........T^{k}}{\Gamma\left(\kappa+1\right)....\Gamma\left(3\right)\Gamma(2-\alpha)^{\kappa-1}} \tag{102}$$

For $H=X-1$, (Appendix)

$$\hat{P}\approx-\frac{2}{\pi}\Sigma_{n=1}^{\infty}\frac{1}{n}Sin\left(n\pi X\right)\Phi^{\alpha}\left(\left(n\pi\right)^2,T\right). \tag{103}$$

Then, the results may be transferred into the initial space, through the transformations,

$$X=\frac{x^{2-\gamma_1}}{\Gamma(3-\gamma_1)},\; T=\frac{t^{2-\gamma_2}}{\Gamma(3-\gamma_2)} \tag{104}$$

$$p\left(x,t\right)=\frac{1}{\Gamma\left(\gamma_1\right)\Gamma(\gamma_2)}\frac{d}{dt}\int_0^t\frac{1}{(t-\tau)^{1-\gamma_1}}\left(\frac{d}{dx}\int_0^x\frac{P(s,\tau)}{\left(x-s\right)^{1-\gamma_2}}ds\right)dt\,. \tag{105}$$

On Λ-fractional geodesics with corners

The presented analysis on fractional hydrodynamics assumes locally stable fields. Nevertheless, Λ-fractional analysis is inherently global, and consequently, non-smooth fields should be accepted. Continuous fields with non-smooth derivatives may be considered in various fields. yielding smooth geodesics (fields), Abraham & Marsden. Nevertheless, continuous fields with possible corner conditions are acceptable, since only globally stable fields with possible non-smooth geodesics are allowed in the Λ-fractional continuum mechanics. The various variational procedures may be globally considered with the consideration of the Erdmann-Weierstrass conditions, Gelfand & Fomin [42-44].

Proceeding to the analysis presented in the preceding paragraph, the balance laws, are described in the general form,

$$\int_{R_t}\left(\frac{\partial}{\partial t}\left(\tilde{n}\ddot{o}\right)-\tilde{n}s\right)dv+\int_{\partial R_t}\left(\tilde{n}\ddot{o}\,v.n-f_{(n)}\right)da-\int_A[\tilde{n}\ddot{o}]\mathbf{V}_{(n)}=0, \tag{106}$$

where Λ is the surface inside the material body with the corners of the geodesics and $\mathbf{V}_{(n)}$ is the normal velocity of the singular surface. Following Chadwick [45-55], the general jump condition is expressed by,

$$\left[\rho V\varphi+f_{(n)}\right]=0, \tag{107}$$

where

$$V=V_n-v.n. \tag{108}$$

The basic jump conditions, corresponding to the basic equation concerning the mass, linear momentum, and energy conservation are expressed by,

$$[\rho V]=0, \tag{109}$$

$$[\rho Vv+t_{(n)}]=0, \tag{110}$$

$$[\rho V(\varepsilon+\frac{1}{2}v.v)+t_{(n)}.v+h_{(n)}]=0. \tag{111}$$

These equations define the shocks in the continuous media. In the fractional analysis, those equations should be satisfied into the Λ-fractional space and the results should be transferred into the initial space.

Fractional shocks in elastic tubes

Let us consider a long cylinder filled with perfect gas and closed at one end by a plane piston. Initially, the gas is considered at rest in the Λ-fractional space with pressure P_o and density R_o. If the piston is moving at a constant speed, U

Citation: Lazopoulos A, Lazopoulos K (2024) On Λ-Fractional fluid mechanics. Ann Math Phys 7(1): 107-117. DOI: https://dx.doi.org/10.17352/amp.000114

determines the speed of propagation of the shock wave in the initial space along with the distributions of the gas density and pressure. Let us point out that a perfect gas is a compressible ideal fluid the pressure being proportional to R^g where g(>1) is a constant.

The shock wave front in the cylinder separates it into two parts, the part with the distributed gas and the part with stationary gas. With the use of Eq.(108), the basic jumping conditions (109-111) are applied to the Λ-fractional space, and if the velocity of the gas behind the piston is U, the mass balance law is expressed by,

$$R(V_n-U)=R_oV_n. \tag{112}$$

Further, the linear momentum balance law is expressed by,

$$P_o-P=-R_oV_nU, \tag{113}$$

and the energy balance,

$$\frac{1}{2}V_n^2+\frac{g}{g-1}\frac{P_o}{R_o}=\frac{1}{2}\left(V_n-U\right)^2+\frac{g}{g-1}\frac{P}{R}, \tag{114}$$

Vn is the local speed of propagation of the shock wave, and P and R are the uniform values of the pressure and density to the rear of the shock. Following Chadwick [45] the shock motion in the Λ-fractional space is defined by the equation

with $c_o=(gP_o/R_o)^{1/2}$.

$$V_n=\frac{1}{4}\left(g+1\right)U+\{\frac{1}{16}\left(g+1\right)^2U^2+c_o^2\}^{1/2}, \tag{115}$$

Finally, the transformation transfers the local shock speed to the initial space.

$$v_n\left(x,t\right)=\frac{1}{\Gamma\left(\gamma_1\right)\Gamma(\gamma_2)}\frac{d}{dt}\int_0^t\frac{1}{(t-\tau)^{1-\gamma_1}}\left(\frac{d}{dx}\int_0^x\frac{V_n}{\left(x-s\right)^{1-\gamma_2}}ds\right)dt. \tag{116}$$

Conclusion

Since the preliminary elements have been defined (Leibniz Fractional Derivative, Fractional Gradient, Fractional Rotation, Fractional Divergence, Fractional Circulation, and Fractional Gauss' Theorem), the basic equations of fluid mechanics are reinstated and analyzed. Further, Fractional Darcy's flow was studied as an application of the fractional flows into porous media. Further, the globally stable flows generate shocks which are described in the context of Λ-fractional analysis. The main issue in our case is the experimental validation of the occurring equations.

(Appendix)

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Citation: Lazopoulos A, Lazopoulos K (2024) On Λ-Fractional fluid mechanics. Ann Math Phys 7(1): 107-117. DOI: https://dx.doi.org/10.17352/amp.000114