Peertechz

Annals of Mathematics and Physics

MATHEMATICS AND PHYSICS GROUP

ANNALS OF Mathematics and Physics OPEN ACCESS JOURNAL

ISSN: 2689-7636 DOI: https://dx.doi.org/10.17352/amp

Research Article

New Quantum Estimates for Midpoint and Trapezoid Type Inequalities Through (α,m)-Convex Functions with Applications

Ghazala Gulshan[1], Muhammad Aamir Ali[2], Hüseyin Budak[3]* and Rashida Hussain[1]

[1]Department of Mathematics, Faculty of Science, Mirpur University of Science and Technology (MUST), Mirpur-10250(AJK), Pakistan

[2]Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210023, China

[3]Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey

Received: 18 July, 2024
Accepted: 03 August, 2024
Published: 06 August, 2024

***Corresponding author:** Hüseyin Budak, Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey,
E-mail: hsyn.budak@gmail.com

ORCiD: https://orcid.org/0000-0001-8843-955X

Keywords: Midpoint inequalities; Trapezoid Inequalities; (α, m)-convex function; Quantum calculus

Abstract

The main goal of current investigation is to present two new q-integral identities for midpoint and trapezoid type inequalities. Then using these identities, we develop several new quantum estimates for midpoint and trapezoid type inequalities via (α, m)-convexity. Some special cases of these new inequalities can be turned into quantum midpoint and quantum trapezoid type inequalities for convex functions, classical midpoint and trapezoid type inequalities for convex functions without having to prove each one separately. Finally, we discuss how the special means can be used to address newly discovered inequalities.

2010 Mathematics Subject Classification. 26D10, 26D15, 26B25.

Introduction

It is well known that modern investigation, directly or indirectly, involves the applications of convexity. Due to its use and significant importance, the concept of convex sets and hence convex functions is largely generalized in various directions. The concept of convexity and its variant forms have played a fundamental role in the development of different fields. Convex functions are powerful tools for proving a large class of inequalities. Today the study of convex functions evolved into a broader theory of functions including quasi-convex functions [1-3], log convex functions [4], co-ordinated convex functions [5,6], harmonically convex functions [7], GA-convex functions [8,9], (α, m) -convex functions [10]. Convexity naturally gives rise to inequalities, Hermite-Hadamard inequalities is the fisrst consequence of convex functions. A function $\mathfrak{F}:\mathfrak{I}\to\mathbb{R}$, where $\mathfrak{I}$ is an interval in $\mathbb{R}$ is called convex, if it satisfies the inequality

$$\mathfrak{F}(\mathfrak{w}\mathfrak{r}+(1-\mathfrak{w})\mathfrak{s})\leq\mathfrak{w}\mathfrak{F}(\mathfrak{r})+(1-\mathfrak{w})\mathfrak{F}(\mathfrak{s})$$

where $\mathfrak{r},\mathfrak{s}\in\mathfrak{I}$ and $\mathfrak{w}\in 0,1]$.

A class of (α,m)-convex functions was introduced by Mihesan and stated as:

Definition 1 *[10] A function $\mathfrak{F}:[0,\mathfrak{y}_2)\to\mathbb{R}$ is called* (α, m)-*convex, if the inequality*

Citation: Gulshan G, Ali MA, Budak H, Hussain R. New Quantum Estimates for Midpoint and Trapezoid Type Inequalities Through (α,m)-Convex Functions with Applications. Ann Math Phys. 2024;7(2):222-231. Available from: https://dx.doi.org/10.17352/amp.000126

$$\mathfrak{F}(\mathfrak{w}\mathfrak{r}+m(1-\mathfrak{w})\mathfrak{s})\leq \mathfrak{w}^{\alpha}\mathfrak{F}(\mathfrak{r})+m(1-\mathfrak{w}^{\alpha})\mathfrak{F}(\mathfrak{s})$$

holds for all $\mathfrak{r},\mathfrak{s}\in[0,\mathfrak{y}_2)$, $\mathfrak{w}\in[0,1],\alpha\in[0,1]$ and $m\in[0,1]$.

It is also well known that $\mathfrak{F}$ is convex if and only if it satisfies the Hermite-Hadamard's inequality, stated below:

$$\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)\leq \frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})d\mathfrak{r}\leq \frac{\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{2}$$

where $\mathfrak{F}:\mathfrak{I}\to\mathbb{R}$ is a convex function and $\mathfrak{y}_1,\mathfrak{y}_2\in\mathfrak{I}$ with $\mathfrak{y}_1<\mathfrak{y}_2$ Convexity is mixed with other mathematical concepts like; optimization [11], time scale [12,13], quantum and post quantum calculus [14].

On the other hand, several works in the field of q-analysis are being carried out, beginning with Euler, in order to achieve mastery in the mathematics that drives quantum computing. Q-calculus is the connection between physics and mathematics. It has a wide range of applications in many fields, e.g., mathematics, including number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions, and other disciplines, as well as mechanics, theory of relativity, and quantum theory [15,16]. q-calculus also has many applications in quantum information theory, which is an interdisciplinary area that surrounds computer science, information theory, philosophy, and cryptography, among other areas [17, 18]. Euler is the inventor of this significant branch of mathematics. Newton used the q -parameter in his work on infinite series. The q -calculus that is known without limits calculus was presented by Jackson [19] in a systematic manner. In 1966, Al-Salam [20] introduced a q -analogue of the q -fractional integrals and q -Riemann-Liouville fractional. Since then, realted research has been increasing gently. In particular, in 2013, Tariboon and Ntouyas introduced the left quantum difference operator and left quantum integral in [21]. In 2020, Bermudo et al. introduced the notion of right quantum derivative and right quantum integral in [22].

Many integrals have also been investigated using quantum and post quantum calculus for different types of functions. For example, in [14,22-30], the authors proved Hermite-Hadamard integral inequalities and their left-right estimates for convex and co-ordinated convex functions by using the quantum derivative and integrals. In [31], the generalized version of q-integral inequalities was presented by Noor et al. In [32] Nwaeze et al. proved certain partametrized quantum integral inequalities for generalized quasi convex functions. Khan et al.proved Hermite-Hadamard inequality using the green function in [33]. For convex and co-ordinated convex functions, Budak et al. [34], Ali et al. [35,36] and Vivas-Cortez et al. [37] developed new quantum Simpson's and Newton's type inequalities.For quantum Ostrowski's type inequalities for convex and co-ordinated convex functions, please refer to [38-40].

Inspired by the ongoing studies, we derive some new inequalities of Midpoint and Trapezoid type inequalities for (α,m) -convex functions by utilizing quantum calculus. The fundamental benefit of these inequalities can be turned into quantum Midpoint and trapezoid type inequalities for convex functions [14,41], classical Midpoint for convex functions [42] and the classical Trapezoid type inequalities for convex functions [43] without having to prove each one separately.

This paper is summarized as follows: Section 2 provides a brief overview of the fundamentals of q-calculus as well as other related studies in this field. In Section 3, we establish two pivotal identities that play a major role in establishing the main outcomes of this paper. The Midpoint and Trapezoid type inequalities for q-differentiable functions via (α, m) -convexity are presented in section 4 and section 5. The special means are described in section 6. The connection between the findings reported here and similar findings in the literature are also taken into account. Section 7 concludes with some suggestions for future research.

Preliminaries and definitions of *q*-calculus

In this section, we first present the definitions and some properties of quantum integrals. We also mention some well known inequalities for quantum integrals. Throughout this paper, let $0<q<1$ be a constant.

The q-number or q -analogue of $n\in\mathbb{N}$ is given by

$$n]_q=\frac{1-q^n}{1-q}=1+q+q^2+\ldots+q^{n-1}.$$

Jackson derived the q -Jackson integral in [44] from 0 to $\mathfrak{y}_2$ as follows:

$$\int_0^{\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})d_q\mathfrak{r}=(1-q)\mathfrak{y}_2\sum_{n=0}^{\infty}q^n\mathfrak{F}(\mathfrak{y}_2q^n)$$

provided the sum converges absolutely.

The q Jackson integral in a generic interval $[\mathfrak{y}_1,\mathfrak{y}_2]$ was given by in [19] and defined as follows:

$$\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})d_q\mathfrak{r}=\int_0^{\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})d_q\mathfrak{r}-\int_0^{\mathfrak{y}_1}\mathfrak{F}(\mathfrak{r})d_q\mathfrak{r}.$$

Definition 2 *[21] Let $\mathfrak{F}:\mathfrak{I}\to\mathbb{R}$ be a continuous function and let $\mathfrak{r}\in\mathfrak{I}$. Then the $q_{\mathfrak{y}_1}$ derivative on $\mathfrak{I}$ of $\mathfrak{F}$ at $\mathfrak{r}$ is defined as*

$$_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})=\frac{\mathfrak{F}(\mathfrak{r})-\mathfrak{F}(q\mathfrak{r}+(1-q)\mathfrak{y}_1)}{(1-q)(\mathfrak{r}-\mathfrak{y}_1)}\quad \mathfrak{r}\neq\mathfrak{y}_1, \tag{1}$$

$$_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)=\lim_{\mathfrak{r}\to\mathfrak{y}_1}{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r}).$$

Definition 3 *If $\mathfrak{y}_1=0$ in (1), then we get classical q -derivative of $\mathfrak{F}(\mathfrak{r})$ at $\mathfrak{r}\in\mathfrak{I}$, given by*

$$_0D_q\mathfrak{F}(\mathfrak{r})=D_q\mathfrak{F}(\mathfrak{r})=\frac{\mathfrak{F}(\mathfrak{r})-\mathfrak{F}(q\mathfrak{r})}{(1-q)\mathfrak{r}}.$$

Definition 4 *[22] Let $\mathfrak{F}:\mathfrak{I}\to\mathbb{R}$ be a continuous function and let $\mathfrak{r}\in\mathfrak{I}$. Then the $q^{\mathfrak{y}_2}$ derivative on $\mathfrak{I}$ of $\mathfrak{F}$ at $\mathfrak{r}$ is defined as*

$$^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})=\frac{\mathfrak{F}(\mathfrak{r})-\mathfrak{F}(q\mathfrak{r}+(1-q)\mathfrak{y}_2)}{(1-q)(\mathfrak{r}-\mathfrak{y}_2)}\quad \mathfrak{r}\neq\mathfrak{y}_2,$$

Citation: Gulshan G, Ali MA, Budak H, Hussain R. New Quantum Estimates for Midpoint and Trapezoid Type Inequalities Through (α,m)-Convex Functions with Applications. Ann Math Phys. 2024;7(2):222-231. Available from: https://dx.doi.org/10.17352/amp.000126

$$^{\eta_2}D_q\mathfrak{F}(\eta_2)=\lim_{\mathfrak{r}\to\eta_2}{}^{\eta_2}D_q\mathfrak{F}(\mathfrak{r}).$$

Definition 5 *[21] Let $\mathfrak{F}:\mathfrak{I}\to\mathbb{R}$ be a continuous function. Then the q_{η_1} -integral on $\mathfrak{I}$ is defined as*

$$\int_{\eta_1}^{\mathfrak{r}}\mathfrak{F}(\mathfrak{w})_{\eta_1}d_q\mathfrak{w}=(1-q)(\mathfrak{r}-\eta_1)\sum_{n=0}^{\infty}q^n\mathfrak{F}(q^n\mathfrak{r}+(1-q^n)\eta_1) \tag{2}$$

for $\mathfrak{r}\in\mathfrak{I}$. If $\eta_1=0$ in (2), then

$$\int_0^{\mathfrak{r}}\mathfrak{F}(\mathfrak{w})_0 d_q\mathfrak{w}=\int_0^{\mathfrak{r}}\mathfrak{F}(\mathfrak{w})d_q\mathfrak{w},$$

where $\int_0^{\mathfrak{r}}\mathfrak{F}(\mathfrak{w})d_q\mathfrak{w}$ is familiar q -definite integral on $[0,\mathfrak{r}]$ defined by the expression

$$\int_0^{\mathfrak{r}}\mathfrak{F}(\mathfrak{w})_0 d_q\mathfrak{w}=\int_0^{\mathfrak{r}}\mathfrak{F}(\mathfrak{w})d_q\mathfrak{w}=(1-q)\mathfrak{r}\sum_{n=0}^{\infty}q^n\mathfrak{F}(q^n\mathfrak{r}).$$

Moreover, if $c\in(\eta_1,\mathfrak{r})$, then the q -integral on $\mathfrak{I}$ is defined as

$$\int_c^{\mathfrak{r}}\mathfrak{F}(\mathfrak{w})_{\eta_1}d_q\mathfrak{w}=\int_{\eta_1}^{\mathfrak{r}}\mathfrak{F}(\mathfrak{w})_{\eta_1}d_q\mathfrak{w}-\int_{\eta_1}^{c}\mathfrak{F}(\mathfrak{w})_{\eta_1}d_q\mathfrak{w}.$$

Definition 6 *[22] Let $\mathfrak{F}:\mathfrak{I}\to\mathbb{R}$ be a continuous function. Then the q^{η_2} -integral on $\mathfrak{I}$ is defined as*

$$\int_{\mathfrak{r}}^{\eta_2}\mathfrak{F}(\mathfrak{w})^{\eta_2}d_q\mathfrak{w}=(1-q)(\eta_2-\mathfrak{r})\sum_{n=0}^{\infty}q^n\mathfrak{F}(q^n\mathfrak{r}+(1-q^n)\eta_2) \tag{3}$$

for $\mathfrak{r}\in\mathfrak{I}$. If $\eta_2=1$ in (3), then

$$\int_{\mathfrak{r}}^{1}\mathfrak{F}(\mathfrak{w})^{1}d_q\mathfrak{w}=\int_{\mathfrak{r}}^{1}\mathfrak{F}(\mathfrak{w})d_q\mathfrak{w},$$

where $\int_0^{\mathfrak{r}}\mathfrak{F}(\mathfrak{w})d_q\mathfrak{w}$ is familiar q -definite integral on $[0,\mathfrak{r}]$ defined by the expression

$$\int_0^{\mathfrak{r}}\mathfrak{F}(\mathfrak{w})_0 d_q\mathfrak{w}=\int_0^{\mathfrak{r}}\mathfrak{F}(\mathfrak{w})d_q\mathfrak{w}=(1-q)\mathfrak{r}\sum_{n=0}^{\infty}q^n\mathfrak{F}(q^n\mathfrak{r}).$$

Moreover, if $c\in(\eta_1,\mathfrak{r})$, then the q -integral on $\mathfrak{I}$ is defined as

$$\int_c^{\mathfrak{r}}\mathfrak{F}(\mathfrak{w})_{\eta_1}d_q\mathfrak{w}=\int_{\eta_1}^{\mathfrak{r}}\mathfrak{F}(\mathfrak{w})_{\eta_1}d_q\mathfrak{w}-\int_{\eta_1}^{c}\mathfrak{F}(\mathfrak{w})_{\eta_1}d_q\mathfrak{w}.$$

n [14], Alp et al. proved the corresponding Hermite-Hadamard inequalities for convex functions by using q_{η_1} -integrals, as follows:

Theorem 1 *[14] If $\mathfrak{F}:[\eta_1,\eta_2]\to\mathbb{R}$ be a convex differentiable function on $[\eta_1,\eta_2]$ and 0<q<1. Then, q_{η_1} -Hermite-Hadamard inequalities*

$$\mathfrak{F}\left(\frac{q\eta_1+\eta_2}{[2]_q}\right)\le\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{w})_{\eta_1}d_q\mathfrak{w}\le\frac{q\mathfrak{F}(\eta_1)+\mathfrak{F}(\eta_2)}{[2]_q}. \tag{4}$$

Bermudo et al. proved the corresponding Hermite-Hadamard inequalities for convex functions by using q^{η_2} -integrals, as follows:

Theorem 2 *[22] If $\mathfrak{F}:[\eta_1,\eta_2]\to\mathbb{R}$ be a convex differentiable function on $[\eta_1,\eta_2]$ and 0<q<1. Then, q^{η_2} -Hermite-Hadamard inequalities*

$$\mathfrak{F}\left(\frac{\eta_1+q\eta_2}{[2]_q}\right)\le\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{w})^{\eta_2}d_q\mathfrak{w}\le\frac{\mathfrak{F}(\eta_1)+q\mathfrak{F}(\eta_2)}{[2]_q}. \tag{5}$$

From Theorem 1 and Theorem 2, one can write the following inequalities:

Corollary 1 *[22] for any convex function $\mathfrak{F}:[\eta_1,\eta_2]\to\mathbb{R}$ and 0<q<1, we have*

$$\mathfrak{F}\left(\frac{q\eta_1+\eta_2}{[2]_q}\right)+\mathfrak{F}\left(\frac{\eta_1+q\eta_2}{[2]_q}\right)\le\frac{1}{\eta_2-\eta_1}\left\{\int_{\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{r})_{\eta_1}d_q\mathfrak{r}+\int_{\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{r})^{\eta_2}d_q\mathfrak{r}\right\}\le\mathfrak{F}(\eta_1)+\mathfrak{F}(\eta_2)$$

and

$$\mathfrak{F}\left(\frac{\eta_1+\eta_2}{2}\right)\le\frac{1}{2(\eta_2-\eta_1)}\left\{\int_{\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{r})_{\eta_1}d_q\mathfrak{r}+\int_{\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{r})^{\eta_2}d_q\mathfrak{r}\right\}\le\frac{\mathfrak{F}(\eta_1)+\mathfrak{F}(\eta_2)}{2}.$$

Theorem 3 *If $\mathfrak{F}:[\eta_1,\eta_2]\to\mathbb{R}$ is a continious function and $z\in\eta_1,\eta_2]$, then the following identities hold:*

$$(i)\ {}_{\eta_1}D_q\int_{\eta_1}^{z}\mathfrak{F}(\mathfrak{r})_{\eta_1}d_q\mathfrak{r}=\mathfrak{F}(z);$$

$$(ii)\int_c^{z}\mathfrak{F}(\mathfrak{r})_{\eta_1}d_q\mathfrak{r}=\mathfrak{F}(z)-\mathfrak{F}(c)\text{ for }c\in(\eta_1,z).$$

Lemma 1 *[45]For continious functions $\mathfrak{F},g:[\eta_1,\eta_2]\to\mathbb{R}$, the following equality true:*

$$\int_0^c g(\mathfrak{w})_{\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+(1-\mathfrak{w})\eta_1 d_q\mathfrak{w}$$

$$=\frac{g(\mathfrak{w})\mathfrak{F}(\mathfrak{w}\eta_2+(1-\mathfrak{w})\eta_1}{\eta_2-\eta_1}\Bigg|_0^c-\frac{1}{\eta_2-\eta_1}\int_0^c D_q g(\mathfrak{w})\mathfrak{F}(q\mathfrak{w}\eta_2+(1-q\mathfrak{w})\eta_1)d_q\mathfrak{w}.$$

Key Identities

In this section, we establish two quantum integral identities using the integration by parts method for quantum integrals to obtain the main outcomes.

Lemma 2 *For a q-differentiable function $\mathfrak{F}:[\eta_1,\eta_2]\to\mathbb{R}$ with ${}_{m\eta_1}D_q\mathfrak{F}$ is continuous and integrable on $[\eta_1,\eta_2]$, the following identity holds:*

$$q(\eta_2-m\eta_1)\left[\int_0^{\frac{1}{[2]_q}}\mathfrak{w}_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)d_q\mathfrak{w}\right.$$

$$\left.+\int_{\frac{1}{[2]_q}}^{1}(\mathfrak{w}-\frac{1}{q})_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)d_q\mathfrak{w}\right]$$

$$=\mathfrak{F}\left(\frac{\eta_2+qm\eta_1}{[2]_q}\right)-\frac{1}{\eta_2-m\eta_1}\int_{m\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{r})_{m\eta_1}d_q\mathfrak{r}. \tag{6}$$

Proof. From fundamental properties of quantum integrals, we have

$$\left[\int_0^{\frac{1}{[2]_q}}\mathfrak{w}_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)d_q\mathfrak{w}\right.$$

Citation: Gulshan G, Ali MA, Budak H, Hussain R. New Quantum Estimates for Midpoint and Trapezoid Type Inequalities Through (α,m)-Convex Functions with Applications. Ann Math Phys. 2024;7(2):222-231. Available from: https://dx.doi.org/10.17352/amp.000126

$$+\int_{\frac{1}{[2]_q}}^{1}\Big(\mathfrak{w}-\frac{1}{q}\Big)_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)d_q\mathfrak{w}\Bigg]$$

$$=\Bigg[\int_0^{\frac{1}{[2]_q}}\mathfrak{w}_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)d_q\mathfrak{w}$$

$$+\int_0^1\Big(\mathfrak{w}-\frac{1}{q}\Big)_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)d_q\mathfrak{w}$$

$$-\int_0^{\frac{1}{[2]_q}}\Big(\mathfrak{w}-\frac{1}{q}\Big)_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)d_q\mathfrak{w}\Bigg]$$

$$=\mathfrak{I}_1+\mathfrak{I}_2-\mathfrak{I}_3.$$

Using the Lemma 1, we have

$$\mathfrak{I}_1=\int_0^{\frac{1}{[2]_q}}\mathfrak{w}_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)d_q\mathfrak{w}$$

$$=\mathfrak{w}\frac{\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)}{\eta_2-m\eta_1}\Bigg|_0^{\frac{1}{[2]_q}}-\frac{1}{\eta_2-m\eta_1}\int_0^{\frac{1}{[2]_q}}\mathfrak{F}(\mathfrak{w}q\eta_2+m(1-\mathfrak{w}q)\eta_1)d_q\mathfrak{w}$$

$$=\frac{1}{[2]_q(\eta_2-m\eta_1)}\mathfrak{F}\left(\frac{\eta_2+qm\eta_1}{[2]_q}\right)$$

$$-\frac{1}{(\eta_2-m\eta_1)}\int_0^{\frac{1}{[2]_q}}\mathfrak{F}(\mathfrak{w}q\eta_2+m(1-\mathfrak{w}q)\eta_1)d_q\mathfrak{w}. \tag{7}$$

Similarly, we have

$$\mathfrak{I}_2=\int_0^1\Big(\mathfrak{w}-\frac{1}{q}\Big)_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)d_q\mathfrak{w}$$

$$=\frac{q-1}{q(\eta_2-m\eta_1)}\mathfrak{F}(\eta_2)+\frac{1}{q(\eta_2-m\eta_1)}\mathfrak{F}(m\eta_1)$$

$$-\frac{1}{\eta_2-m\eta_1}\int_0^1\mathfrak{F}(\mathfrak{w}q\eta_2+m(1-\mathfrak{w}q)\eta_1)d_q\mathfrak{w}$$

$$=\frac{q-1}{q(\eta_2-m\eta_1)}\mathfrak{F}(\eta_2)+\frac{1}{q(\eta_2-m\eta_1)}\mathfrak{F}(m\eta_1)$$

$$-\frac{1}{q(\eta_2-m\eta_1)^2}\int_{m\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{r})_{m\eta_1}d_q\mathfrak{r}+\frac{1-q}{q(\eta_2-m\eta_1)}\mathfrak{F}(\eta_2) \tag{8}$$

and

$$\mathfrak{I}_3=\int_0^{\frac{1}{[2]_q}}\Big(\mathfrak{w}-\frac{1}{q}\Big)_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)d_q\mathfrak{w}$$

$$=-\frac{1}{q[2]_q(\eta_2-m\eta_1)}\mathfrak{F}\Big(\frac{\eta_2+qm\eta_1}{[2]_q}\Big)+\frac{1}{q(\eta_2-m\eta_1)}\mathfrak{F}(m\eta_1)$$

$$-\frac{1}{\eta_2-m\eta_1}\int_0^{\frac{1}{[2]_q}}\mathfrak{F}(\mathfrak{w}q\eta_2+m(1-\mathfrak{w}q)\eta_1)d_q\mathfrak{w}. \tag{9}$$

Thus from (7), (8) and (9), we have

$$\mathfrak{I}_1+\mathfrak{I}_2-\mathfrak{I}_3=\frac{1}{q(\eta_2-m\eta_1)}\mathfrak{F}\Big(\frac{\eta_2+qm\eta_1}{[2]_q}\Big)-\frac{1}{q(\eta_2-m\eta_1)^2}\int_{m\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{r})_{m\eta_1}d_q\mathfrak{r} \tag{10}$$

and we obtain required equality (6) by multiplying $q(\eta_2-m\eta_1)$ on both sides of (10). Thus, the proof is accomplished.

Remark 1 *In Lemma 2, we have*

- if we set $\alpha=m=1$, then we find [14, Lemma 11].
- If we set $\alpha=m=1$ and later taking $q\to 1^-$, then we find [42, Lemma 2.1].

Lemma 3 *For a q-differentiable function $\mathfrak{F}:[\eta_1,\eta_2]\to\mathbb{R}$ with ${}_{m\eta_1}D_q\mathfrak{F}$ is continious and integrable on $[\eta_1,\eta_2]$, the following equality holds:*

$$\frac{1}{\eta_2-m\eta_1}\int_{m\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{r})_{m\eta_1}d_q\mathfrak{r}-\frac{\mathfrak{F}(\eta_2)+q\mathfrak{F}(m\eta_1)}{[2]_q}$$

$$=\frac{q(\eta_2-m\eta_1)}{[2]_q}\int_0^1(1-[2]_q\mathfrak{w})\,{}_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)d_q\mathfrak{w} \tag{11}$$

Proof. From fundamental properties of quantum integral, we have

$$\int_0^1(1-[2]_q\mathfrak{w})\,{}_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)d_q\mathfrak{w}$$

$$=\frac{(1-[2]_q\mathfrak{w})\mathfrak{F}(\eta_2\mathfrak{w}+m(1-\mathfrak{w})\eta_1)}{\eta_2-m\eta_1}\Bigg|_0^1+\frac{[2]_q}{\eta_2-m\eta_1}\int_0^1\mathfrak{F}(q\mathfrak{w}\eta_2+m(1-q\mathfrak{w})\eta_1)d_q\mathfrak{w}$$

$$=-\frac{q\mathfrak{F}(\eta_2)+\mathfrak{F}(m\eta_1)}{\eta_2-m\eta_1}+\frac{[2]_q}{\eta_2-m\eta_1}\int_0^1\mathfrak{F}(q\mathfrak{w}\eta_2+m(1-q\mathfrak{w})\eta_1)d_q\mathfrak{w}$$

$$=-\frac{q\mathfrak{F}(\eta_2)+\mathfrak{F}(m\eta_1)}{\eta_2-m\eta_1}+\frac{[2]_q(1-q)}{q(\eta_2-m\eta_1)}\sum_{n=0}^{\infty}q^{n+1}\mathfrak{F}(q^{n+1}\eta_2+m(1-q^{n+1})\eta_1)$$

$$=-\frac{q\mathfrak{F}(\eta_2)+\mathfrak{F}(m\eta_1)}{\eta_2-m\eta_1}+\frac{[2]_q(1-q)}{q(\eta_2-m\eta_1)}\left(\sum_{k=0}^{\infty}q^k\mathfrak{F}(q^k\eta_2+m(1-q^k)\eta_1)-\mathfrak{F}(\eta_2)\right)$$

$$=-\frac{q\mathfrak{F}(\eta_2)+\mathfrak{F}(m\eta_1)}{\eta_2-m\eta_1}+\frac{[2]_q}{q(\eta_2-m\eta_1)^2}\int_{m\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{r})_{m\eta_1}d_q\mathfrak{r}-\frac{[2]_q(1-q)}{q(\eta_2-m\eta_1)}\mathfrak{F}(\eta_2)$$

$$=-\frac{\mathfrak{F}(\eta_2)+q\mathfrak{F}(m\eta_1)}{q(\eta_2-m\eta_1)}+\frac{[2]_q}{q(\eta_2-m\eta_1)^2}\int_{m\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{r})_{m\eta_1}d_q\mathfrak{r}. \tag{12}$$

and we obtain the required equality (11) by multiplying $\frac{q(\eta_2-m\eta_1)}{[2]_q}$ on both sides of (12). Thus, the proof is accomplished.

Remark 2 *In Lemma 3, we have*

- If we set $\alpha=m=1$, then we find [41, Lemma 3.1].
- If we set $\alpha=m=1$ and later taking limit as $q\to 1^-$, then we find [43, Lemma 2.1].

Midpoint Type inequalities for (α, *m*) -convex functions

In this section, we prove Midpoint type inequalities for differentiable (α, m)-convex functions.

Citation: Gulshan G, Ali MA, Budak H, Hussain R. New Quantum Estimates for Midpoint and Trapezoid Type Inequalities Through (α,m)-Convex Functions with Applications. Ann Math Phys. 2024;7(2):222-231. Available from: https://dx.doi.org/10.17352/amp.000126

Theorem 4 *Under the assumption of Lemma 2, if* $\left|{}_{m\eta_1}D_q\mathfrak{F}\right|$ *is* (α,m) *convex function over* $[\eta_1,\eta_2]$, *then we have the following midpoint type inequality:*

$$\left|\mathfrak{F}\left(\frac{\eta_2+qm\eta_1}{[2]_q}\right)-\frac{1}{\eta_2-m\eta_1}\int_{m\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{r})\,{}_{m\eta_1}d_q\mathfrak{r}\right|$$

$$\leq q(\eta_2-m\eta_1)[(A_1(q)+A_3(q))\left|{}_{m\eta_1}D_q\mathfrak{F}(\eta_2)\right|$$

$$+m(A_2(q)+A_4(q))\left|{}_{m\eta_1}D_q\mathfrak{F}(\eta_1)\right|], \tag{13}$$

where

$$A_1(q)=\int_0^{\frac{1}{[2]_q}}\mathfrak{w}^{\alpha+1}d_q\mathfrak{w}=\frac{1}{[2]_q^{\alpha+2}[\alpha+2]_q}$$

$$A_2(q)=\int_0^{\frac{1}{[2]_q}}\mathfrak{w}(1-\mathfrak{w}^{\alpha})\,d_q\mathfrak{w}=\frac{1}{[2]_q^3}-\frac{1}{[2]_q^{\alpha+2}[\alpha+2]_q}$$

$$A_3(q)=\int_{\frac{1}{[2]_q}}^1\mathfrak{w}^{\alpha}\left(\frac{1}{q}-\mathfrak{w}\right)d_q\mathfrak{w}=\frac{1}{q[\alpha+1]_q}-\frac{1}{[\alpha+2]_q}$$

$$-\frac{1}{q[2]_q^{\alpha+1}[\alpha+1]_q}+\frac{1}{[2]_q^{\alpha+2}[\alpha+2]_q}$$

$$A_4(q)=\int_{\frac{1}{[2]_q}}^1\left(\frac{1}{q}-\mathfrak{w}\right)(1-\mathfrak{w}^{\alpha})\,d_q\mathfrak{w}=\frac{1}{[2]_q^3}-\frac{1}{q[\alpha+1]_q}+\frac{1}{[\alpha+2]_q}$$

$$+\frac{1}{q[2]_q^{\alpha+1}[\alpha+1]_q}-\frac{1}{[2]_q^{\alpha+2}[\alpha+2]_q}.$$

Proof. By taking modulus in (6), and using (α, m) -convexity of $\left|{}_{m\eta_1}D_q\mathfrak{F}\right|$, we have

$$\left|\mathfrak{F}\left(\frac{\eta_2+qm\eta_1}{[2]_q}\right)-\frac{1}{\eta_2-m\eta_1}\int_{m\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{r})\,{}_{m\eta_1}d_q\mathfrak{r}\right|$$

$$\leq q(\eta_2-m\eta_1)\left[\int_0^{\frac{1}{[2]_q}}\mathfrak{w}\left|{}_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)\right|d_q\mathfrak{w}\right.$$

$$\left.+\int_{\frac{1}{[2]_q}}^1\left(\frac{1}{q}-\mathfrak{w}\right)\left|{}_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)\right|d_q\mathfrak{w}\right]$$

$$\leq\int_0^{\frac{1}{[2]_q}}\mathfrak{w}^{\alpha+1}\left|{}_{m\eta_1}D_q\mathfrak{F}(\eta_2)\right|d_q\mathfrak{w}+\int_0^{\frac{1}{[2]_q}}m(\mathfrak{w}-\mathfrak{w}^{\alpha+1})\left|{}_{m\eta_1}D_q\mathfrak{F}(\eta_1)\right|d_q\mathfrak{w}$$

$$+\int_{\frac{1}{[2]_q}}^1\left(\frac{\mathfrak{w}^{\alpha}}{q}-\mathfrak{w}^{\alpha+1}\right)\left|{}_{m\eta_1}D_q\mathfrak{F}(\eta_2)\right|d_q\mathfrak{w}$$

$$+\int_{\frac{1}{[2]_q}}^1m\left(\frac{1}{q}-\mathfrak{w}\right)(1-\mathfrak{w}^{\alpha})\left|{}_{m\eta_1}D_q\mathfrak{F}(\eta_1)\right|d_q\mathfrak{w}$$

$$=q(\eta_2-m\eta_1)\left[A_1(q)\left|{}_{m\eta_1}D_q\mathfrak{F}(\eta_2)\right|+m\,A_2(q)\left|{}_{m\eta_1}D_q\mathfrak{F}(\eta_1)\right|\right.$$

$$\left.+A_3(q)\left|{}_{m\eta_1}D_q\mathfrak{F}(\eta_2)\right|+m\,A_4(q)\left|{}_{m\eta_1}D_q\mathfrak{F}(\eta_1)\right|\right]$$

$$=q(\eta_2-m\eta_1)[(A_1(q)+A_3(q))\left|{}_{m\eta_1}D_q\mathfrak{F}(\eta_2)\right|+m(A_2(q)+A_4(q))\left|{}_{m\eta_1}D_q\mathfrak{F}(\eta_1)\right|].$$

Thus, the proof is accomplished.

Remark 3 *In Theorem 4, we have*

- If we set $\alpha=m=1$, then we find [14, Theorem 13].
- If we set $\alpha=m=1$ and later taking the limit as $q\to1^-$, then we find [42, Theorem 2.2].

Theorem 5 *Under the assumption of Lemma 2, If* $\left|{}_{m\eta_1}D_q\mathfrak{F}(\mathfrak{r})\right|^r, r\geqslant1$ *is* (α, m) *convex function over* $[\eta_1,\eta_2]$, *then we have the following midpoint type inequality:*

$$\left|\mathfrak{F}\left(\frac{\eta_2+qm\eta_1}{[2]_q}\right)-\frac{1}{\eta_2-m\eta_1}\int_{m\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{r})\,{}_{m\eta_1}d_q\mathfrak{r}\right|$$

$$\leq\frac{q(\eta_2-m\eta_1)}{[2]_q^{\frac{3(r-1)}{r}}}\left[\left(A_1(q)\left|{}_{m\eta_1}D_q\mathfrak{F}(\eta_2)\right|^r+m\,A_2(q)\left|{}_{m\eta_1}D_q\mathfrak{F}(\eta_1)\right|^r\right)^{\frac{1}{r}}\right.$$

$$\left.+\left(A_3(q)\left|{}_{m\eta_1}D_q\mathfrak{F}(\eta_2)\right|^r+m\,A_4(q)\left|{}_{m\eta_1}D_q\mathfrak{F}(\eta_1)\right|^r\right)^{\frac{1}{r}}\right] \tag{14}$$

Proof. By taking modulus in (6), and using power mean inequality, we have

$$\left|\mathfrak{F}\left(\frac{\eta_2+qm\eta_1}{[2]_q}\right)-\frac{1}{\eta_2-m\eta_1}\int_{m\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{r})\,{}_{m\eta_1}d_q\mathfrak{r}\right|$$

$$\leq q(\eta_2-m\eta_1)\left[\int_0^{\frac{1}{[2]_q}}\left|\mathfrak{w}\,{}_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)\right|d_q\mathfrak{w}\right.$$

$$\left.+\int_{\frac{1}{[2]_q}}^1\left|\left(\mathfrak{w}-\frac{1}{q}\right){}_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)\right|d_q\mathfrak{w}\right]$$

$$\leq q(\eta_2-m\eta_1)\left[\left(\int_0^{\frac{1}{[2]_q}}\mathfrak{w}\,d_q\mathfrak{w}\right)^{1-\frac{1}{r}}\left(\int_0^{\frac{1}{[2]_q}}\mathfrak{w}\left|{}_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)\right|^r d_q\mathfrak{w}\right)^{\frac{1}{r}}\right.$$

$$\left.+\left(\int_{\frac{1}{[2]_q}}^1\left(\frac{1}{q}-\mathfrak{w}\right)d_q\mathfrak{w}\right)^{1-\frac{1}{r}}\left(\int_{\frac{1}{[2]_q}}^1\left(\frac{1}{q}-\mathfrak{w}\right)\left|{}_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)\right|^r d_q\mathfrak{w}\right)^{\frac{1}{r}}\right]$$

By applying (α, m)-convexity of $\left|{}_{m\eta_1}D_q\mathfrak{F}(\mathfrak{r})\right|^r$, we have

$$\left|\mathfrak{F}\left(\frac{\eta_2+qm\eta_1}{[2]_q}\right)-\frac{1}{\eta_2-m\eta_1}\int_{m\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{r})\,{}_{m\eta_1}d_q\mathfrak{r}\right|$$

$$\leq q(\eta_2-m\eta_1)\left(\frac{1}{[2]_q^3}\right)^{1-\frac{1}{r}}$$

Citation: Gulshan G, Ali MA, Budak H, Hussain R. New Quantum Estimates for Midpoint and Trapezoid Type Inequalities Through (α,m)-Convex Functions with Applications. Ann Math Phys. 2024;7(2):222-231. Available from: https://dx.doi.org/10.17352/amp.000126

$$\times\left[\left(\int_0^{\frac{1}{[2]_q}}\mathfrak{w}^{\alpha+1}\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^r d_q\mathfrak{w}+\int_0^{\frac{1}{[2]_q}}m\left(\mathfrak{w}-\mathfrak{w}^{\alpha+1}\right)\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^r d_q\mathfrak{w}\right)^{\frac{1}{r}}\right.$$

$$\left(+\int_{\frac{1}{[2]_q}}^1\left(\frac{\mathfrak{w}^\alpha}{q}-\mathfrak{w}^{\alpha+1}\right)\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^r d_q\mathfrak{w}\right.$$

$$\left.\left.+\int_{\frac{1}{[2]_q}}^1 m\left(\frac{1}{q}-\mathfrak{w}\right)(1-\mathfrak{w}^\alpha)\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^r d_q\mathfrak{w}\right)^{\frac{1}{r}}\right]$$

$$=\frac{q(\mathfrak{y}_2-m\mathfrak{y}_1)}{[2]_q^{\frac{3(r-1)}{r}}}\left[\left(A_1(q)\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^r+m\,A_2(q)\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^r\right)^{\frac{1}{r}}\right.$$

$$\left.+\left(A_3(q)\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^r+m\,A_4(q)\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^r\right)^{\frac{1}{r}}\right].$$

Thus, the proof is accomplished.

Remark 4 *In Theorem 5, If we set* $\alpha=m=1$, *then we find [14, Theorem 16].*

Theorem 6 *Under the assumption of Lemma 2, if* $r>1$ *is a real number, if* $\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})\right|^r$ *is* (α,m) *convex function over* $[\mathfrak{y}_1,\mathfrak{y}_2]$, *then we have the following midpoint type inequality,*

$r^{-1}+s^{-1}=1$.

$$\left|\mathfrak{F}\left(\frac{\mathfrak{y}_2+qm\mathfrak{y}_1}{[2]_q}\right)-\frac{1}{\mathfrak{y}_2-m\mathfrak{y}_1}\int_{m\mathfrak{y}_1}^{\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})\,{}_{m\mathfrak{y}_1}d_q\mathfrak{r}\right|$$

$$\leq q(\mathfrak{y}_2-m\mathfrak{y}_1)$$

$$\times\left[\left(\frac{1}{[2]_q^{s+1}[s+1]_q}\right)^{\frac{1}{s}}\left(B_1(q)\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^r+mC_1(q)\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^r\right)^{\frac{1}{r}}\right.$$

$$\left.+(\eta(q))^{\frac{1}{s}}\left(B_2(q)\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^r+mC_2(q)\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^r\right)^{\frac{1}{r}}\right],\qquad(15)$$

where

$$B_1(q)=\int_0^{\frac{1}{[2]_q}}\mathfrak{w}^\alpha d_q\mathfrak{w}=\frac{1}{[2]_q^{\alpha+1}[\alpha+1]_q}$$

$$B_2(q)=\int_{\frac{1}{[2]_q}}^1\mathfrak{w}^\alpha d_q\mathfrak{w}=\frac{1}{[\alpha+1]_q}-\frac{1}{[2]_q^{\alpha+1}[\alpha+1]_q}$$

$$C_1(q)=\int_0^{\frac{1}{[2]_q}}(1-\mathfrak{w}^\alpha)\,d_q\mathfrak{w}=\frac{1}{[2]_q}-\frac{1}{[2]_q^{\alpha+1}[\alpha+1]_q}$$

$$C_2(q)=\int_{\frac{1}{[2]_q}}^1(1-\mathfrak{w}^\alpha)\,d_q\mathfrak{w}=\frac{q}{[2]_q}-\frac{1}{[\alpha+1]_q}+\frac{1}{[2]_q^{\alpha+1}[\alpha+1]_q}$$

$$\eta(q)=\int_{\frac{1}{[2]_q}}^1\left(\frac{1}{q}-\mathfrak{w}\right)^s d_q\mathfrak{w}.$$

Proof. Taking absolute value of (6) and using the Hölder's inequality, we have

$$\left|\mathfrak{F}\left(\frac{\mathfrak{y}_2+qm\mathfrak{y}_1}{[2]_q}\right)-\frac{1}{\mathfrak{y}_2-m\mathfrak{y}_1}\int_{m\mathfrak{y}_1}^{\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})\,{}_{m\mathfrak{y}_1}d_q\mathfrak{r}\right|$$

$$\leq q(\mathfrak{y}_2-m\mathfrak{y}_1)\left[\int_0^{\frac{1}{[2]_q}}\left|\mathfrak{w}\,{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_2+m(1-\mathfrak{w})\mathfrak{y}_1)\right|d_q\mathfrak{w}\right.$$

$$\left.+\int_{\frac{1}{[2]_q}}^1\left|\left(\mathfrak{w}-\frac{1}{q}\right){}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_2+m(1-\mathfrak{w})\mathfrak{y}_1)\right|d_q\mathfrak{w}\right]$$

$$\leq q(\mathfrak{y}_2-m\mathfrak{y}_1)\left[\left(\int_0^{\frac{1}{[2]_q}}\mathfrak{w}^s d_q\mathfrak{w}\right)^{\frac{1}{s}}\left(\int_0^{\frac{1}{[2]_q}}\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_2+m(1-\mathfrak{w})\mathfrak{y}_1)\right|^r d_q\mathfrak{w}\right)^{\frac{1}{r}}\right.$$

$$\left.+\left(\int_{\frac{1}{[2]_q}}^1\left(\frac{1}{q}-\mathfrak{w}\right)^s d_q\mathfrak{w}\right)^{\frac{1}{s}}\left(\int_{\frac{1}{[2]_q}}^1\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{w}\mathfrak{y}_2+m(1-\mathfrak{w})\mathfrak{y}_1)\right|^r d_q\mathfrak{w}\right)^{\frac{1}{r}}\right]$$

By applying (α,m) convexity of $\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})\right|^r$, we have

$$\left|\mathfrak{F}\left(\frac{\mathfrak{y}_2+qm\mathfrak{y}_1}{[2]_q}\right)-\frac{1}{\mathfrak{y}_2-m\mathfrak{y}_1}\int_{m\mathfrak{y}_1}^{\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})\,{}_{m\mathfrak{y}_1}d_q\mathfrak{r}\right|$$

$$\leq q(\mathfrak{y}_2-m\mathfrak{y}_1)\left[\left(\int_0^{\frac{1}{[2]_q}}\mathfrak{w}^s d_q\mathfrak{w}\right)^{\frac{1}{s}}\right.$$

$$\times\left(\int_0^{\frac{1}{[2]_q}}\mathfrak{w}^\alpha\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^r d_q\mathfrak{w}+\int_0^{\frac{1}{[2]_q}}m\,(1-\mathfrak{w}^\alpha)\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^r d_q\mathfrak{w}\right)^{\frac{1}{r}}$$

$$+\left(\int_{\frac{1}{[2]_q}}^1\left(\frac{1}{q}-\mathfrak{w}\right)^s d_q\mathfrak{w}\right)^{\frac{1}{s}}$$

$$\left.\times\left(\int_{\frac{1}{[2]_q}}^1\mathfrak{w}^\alpha\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^r d_q\mathfrak{w}+\int_{\frac{1}{[2]_q}}^1 m\,(1-\mathfrak{w}^\alpha)\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^r d_q\mathfrak{w}\right)^{\frac{1}{r}}\right]$$

$$=q(\mathfrak{y}_2-m\mathfrak{y}_1)$$

$$\times\left[\left(\frac{1}{[2]_q^{s+1}[s+1]_q}\right)^{\frac{1}{s}}\left(B_1(q)\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^r+mC_1(q)\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^r\right)^{\frac{1}{r}}\right.$$

$$\left.+(\eta(q))^{\frac{1}{s}}\left(B_2(q)\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^r+mC_2(q)\left|{}_{m\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^r\right)^{\frac{1}{r}}\right]\qquad(16)$$

Thus, the proof is accomplished.

Remark 5 *In Theorem 6, we have*

- If we set $\alpha=m=1$, then we find [14, Theorem 18].
- If we set $\alpha=m=1$ and later taking the limit as $q\to1^-$, then we find [42, Theorem 2.3].

Citation: Gulshan G, Ali MA, Budak H, Hussain R. New Quantum Estimates for Midpoint and Trapezoid Type Inequalities Through (α,m)-Convex Functions with Applications. Ann Math Phys. 2024;7(2):222-231. Available from: https://dx.doi.org/10.17352/amp.000126

Trapezoid type inequalities for (α, m)-convex functions

In this section, we prove Trapezoid type inequalities for differentiable (α, m) -convex functions.

Theorem 7 *Under the assumption of Lemma 3, if* $|_{m\eta_1}D_q\mathfrak{F}|$ *is* (α, m) *convex function over* $[\eta_1,\eta_2]$, *then we have the following trapezoid type inequality:*

$$\left|-\frac{\mathfrak{F}(\eta_2)+q\mathfrak{F}(m\eta_1)}{[2]_q}+\frac{1}{\eta_2-m\eta_1}\int_{m\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{r})_{m\eta_1}d_q\mathfrak{r}\right|$$

$$\leq\frac{q(\eta_2-m\eta_1)}{[2]_q}\Big[\left|_{m\eta_1}D_q\mathfrak{F}(\eta_2)\right|(K_1(q)-K_2(q))$$

$$+m\left|_{m\eta_1}D_q\mathfrak{F}(\eta_1)\right|(L_1(q)-L_2(q))\Big], \tag{17}$$

where

$$K_1(q)=\int_0^{\frac{1}{[2]_q}}(\mathfrak{w}^\alpha-[2]_q\mathfrak{w}^{\alpha+1})d_q\mathfrak{w}=\frac{q^{\alpha+1}}{[2]_q^{\alpha+1}[\alpha+1]_q[\alpha+2]_q}$$

$$K_2(q)=\int_{\frac{1}{[2]_q}}^1(\mathfrak{w}^\alpha-[2]_q\mathfrak{w}^{\alpha+1})d_q\mathfrak{w}=\frac{1}{[\alpha+1]_q}-\frac{[2]_q}{[\alpha+2]_q}-\frac{q^{\alpha+1}}{[2]_q^{\alpha+1}[\alpha+1]_q[\alpha+2]_q}$$

$$L_1(q)=\int_0^{\frac{1}{[2]_q}}(1-[2]_q\mathfrak{w})(1-\mathfrak{w}^\alpha)d_q\mathfrak{w}=\frac{q}{[2]_q^2}-\frac{q^{\alpha+1}}{[2]_q^{\alpha+1}[\alpha+1]_q[\alpha+2]_q}$$

$$L_2(q)=\int_{\frac{1}{[2]_q}}^1(1-[2]_q\mathfrak{w})(1-\mathfrak{w}^\alpha)d_q\mathfrak{w}$$

$$=\frac{q[\alpha]_q}{[\alpha+1]_q[\alpha+2]_q}-\frac{q}{[2]_q^2}+\frac{1}{[\alpha+1]_q[2]_q^{\alpha+1}}-\frac{1}{[\alpha+2]_q[2]_q^{\alpha+1}}$$

Proof. By taking modulus in (11), and using (α, m) -convexity of $|_{m\eta_1}D_q\mathfrak{F}|$, we have

$$\left|-\frac{\mathfrak{F}(\eta_2)+q\mathfrak{F}(m\eta_1)}{[2]_q}+\frac{1}{\eta_2-m\eta_1}\int_{m\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{r})_{m\eta_1}d_q\mathfrak{r}\right|$$

$$=\frac{q(\eta_2-m\eta_1)}{[2]_q}\left|\int_0^1(1-[2]_q\mathfrak{w})_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)d_q\mathfrak{w}\right|$$

$$\leq\frac{q(\eta_2-m\eta_1)}{[2]_q}\int_0^1\left|(1-[2]_q\mathfrak{w})\right|\left|_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)\right|d_q\mathfrak{w}$$

$$\leq\frac{q(\eta_2-m\eta_1)}{[2]_q}\left(\begin{array}{c}\int_0^1\left|(1-[2]_q\mathfrak{w})\right|\mathfrak{w}^\alpha\left|_{m\eta_1}D_q\mathfrak{F}(\eta_2)\right|d_q\mathfrak{w}\\+\int_0^1\left|(1-[2]_q\mathfrak{w})\right|m(1-\mathfrak{w}^\alpha)\left|_{m\eta_1}D_q\mathfrak{F}(\eta_1)\right|d_q\mathfrak{w}\end{array}\right)$$

$$=\frac{q(\eta_2-m\eta_1)}{[2]_q}\Big[\left|_{m\eta_1}D_q\mathfrak{F}(\eta_2)\right|(K_1(q)-K_2(q))+m\left|_{m\eta_1}D_q\mathfrak{F}(\eta_1)\right|(L_1(q)-L_2(q))\Big]$$

Thus, the proof is accomplished.

Remark 6 *In Theorem 7, we have*

- If we set $\alpha=m=1$, then we find [41, Theorem 4.1].
- If we set $\alpha=m=1$ and later taking the limit as $q\to1^-$, then we find [43, Theorem 2.2].

Theorem 8 *Under the assumption of Lemma 3, if* $|_{m\eta_1}D_q\mathfrak{F}(\mathfrak{r})|^r, r\geq1$ *is* (α, m) *convex function over* $[\eta_1,\eta_2]$, *then we have the following trapezoid type inequality:*

$$\left|-\frac{\mathfrak{F}(\eta_2)+q\mathfrak{F}(m\eta_1)}{[2]_q}+\frac{1}{\eta_2-m\eta_1}\int_{m\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{r})_{m\eta_1}d_q\mathfrak{r}\right|$$

$$\leq\frac{q(\eta_2-m\eta_1)}{[2]_q}\left(\frac{2q}{[2]_q^2}\right)^{1-\frac{1}{r}}$$

$$\times\Big[\left|_{m\eta_1}D_q\mathfrak{F}(\eta_2)\right|^r(K_1(q)-K_2(q))+m\left|_{m\eta_1}D_q\mathfrak{F}(\eta_1)\right|^r(L_1(q)-L_2(q))\Big], \tag{18}$$

Proof. By taking modulus in (11) and using power mean inequality, we have

$$\left|-\frac{\mathfrak{F}(\eta_2)+q\mathfrak{F}(m\eta_1)}{[2]_q}+\frac{1}{\eta_2-m\eta_1}\int_{m\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{r})_{m\eta_1}d_q\mathfrak{r}\right|$$

$$=\frac{q(\eta_2-m\eta_1)}{[2]_q}\left|\int_0^1(1-[2]_q\mathfrak{w})_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)d_q\mathfrak{w}\right|$$

$$\leq\frac{q(\eta_2-m\eta_1)}{[2]_q}\left(\int_0^1\left|(1-[2]_q\mathfrak{w})\right|d_q\mathfrak{w}\right)^{1-\frac{1}{r}}$$

$$\times\left(\int_0^1\left|(1-[2]_q\mathfrak{w})\right|\left|_{m\eta_1}D_q\mathfrak{F}(\mathfrak{w}\eta_2+m(1-\mathfrak{w})\eta_1)\right|^rd_q\mathfrak{w}\right)^{\frac{1}{r}}$$

By applying (α, m)-convexity of $|_{m\eta_1}D_q\mathfrak{F}(\mathfrak{r})|^r$, we have

$$\left|-\frac{\mathfrak{F}(\eta_2)+q\mathfrak{F}(m\eta_1)}{[2]_q}+\frac{1}{\eta_2-m\eta_1}\int_{m\eta_1}^{\eta_2}\mathfrak{F}(\mathfrak{r})_{m\eta_1}d_q\mathfrak{r}\right|$$

$$\leq\frac{q(\eta_2-m\eta_1)}{[2]_q}\left(\int_0^1\left|(1-[2]_q\mathfrak{w})\right|d_q\mathfrak{w}\right)^{1-\frac{1}{r}}$$

$$\times\left(\begin{array}{c}\int_0^1\left|(1-[2]_q\mathfrak{w})\right|\mathfrak{w}^\alpha\left|_{m\eta_1}D_q\mathfrak{F}(\eta_2)\right|^rd_q\mathfrak{w}\\+\int_0^1\left|(1-[2]_q\mathfrak{w})\right|m(1-\mathfrak{w}^\alpha)\left|_{m\eta_1}D_q\mathfrak{F}(\eta_1)\right|^rd_q\mathfrak{w}\end{array}\right)^{\frac{1}{r}}$$

$$=\frac{q(\eta_2-m\eta_1)}{[2]_q}\left(\frac{2q}{[2]_q^2}\right)^{1-\frac{1}{r}}$$

$$\times\Big[\left|_{m\eta_1}D_q\mathfrak{F}(\eta_2)\right|^r(K_1(q)-K_2(q))+m\left|_{m\eta_1}D_q\mathfrak{F}(\eta_1)\right|^r(L_1(q)-L_2(q))\Big]$$

Thus, the proof is accomplished.

Remark 7 *In Theorem 8, we have*

- If we set $\alpha=m=1$, then we find [41, Theorem 4.2].

Citation: Gulshan G, Ali MA, Budak H, Hussain R. New Quantum Estimates for Midpoint and Trapezoid Type Inequalities Through (α,m)-Convex Functions with Applications. Ann Math Phys. 2024;7(2):222-231. Available from: https://dx.doi.org/10.17352/amp.000126

- If we set $\alpha = m = 1$ and later taking the limit as $q \to 1^-$, then we find [46, Theorem 6].

Application to special means

For any positive number $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{R}$, we consider the following means:

- The Arithmetic mean

$$A(\mathfrak{y}_1, \mathfrak{y}_2) = \frac{\mathfrak{y}_1 + \mathfrak{y}_2}{2}$$

- The Harmonic mean

$$H(\mathfrak{y}_1, \mathfrak{y}_2) = \frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1 + \mathfrak{y}_2}$$

- The Geometric mean

$$G(\mathfrak{y}_1, \mathfrak{y}_2) = \sqrt{\mathfrak{y}_1\mathfrak{y}_2}\ .$$

Proposition 1 *Let* $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{R}, \mathfrak{y}_1 < \mathfrak{y}_2, \alpha \in 0,1], m \in 0,1]$ *and* $0 < q < 1$. *Then we have*

$$\left| H(\mathfrak{y}_2, qm\mathfrak{y}_1) - \frac{G^2(\mathfrak{y}_2, qm\mathfrak{y}_1)}{A(1,q)} \Upsilon_1 \right|$$

$$\leq \frac{q(\mathfrak{y}_2 - m\mathfrak{y}_1)G^2(\mathfrak{y}_2, qm\mathfrak{y}_1)}{A(1,q)}$$

$$\times \frac{1}{\mathfrak{y}_2(q\mathfrak{y}_2 + (1-q)m\mathfrak{y}_1)}(A_1(q) + A_3(q)) + \frac{1}{m\mathfrak{y}_1^2}(A_2(q) + A_4(q)) \tag{19}$$

where $\Upsilon_1 = \frac{1}{(\mathfrak{y}_2 - m\mathfrak{y}_1)} \int_{m\mathfrak{y}_1}^{\mathfrak{y}_2} \frac{1}{\mathfrak{r}}\ {}_{m\mathfrak{y}_1}d_q\mathfrak{r} = (1-q)\sum_{n=0}^{\infty} \frac{q^n}{q^n\mathfrak{y}_2 + m(1-q^n)\mathfrak{y}_1}$.

Proof. The inequality (13) for function $\mathfrak{F}(\mathfrak{r}) = \frac{2q\mathfrak{y}_2 m\mathfrak{y}_1}{[2]_q \mathfrak{r}}$ leads to required result. If we take $\mathfrak{y}_1 = 1, \mathfrak{y}_2 = 2, q = 0.5, m = 0.5$ and $\alpha = 0.5$ in (19), we get

$$\left| H(\mathfrak{y}_2, qm\mathfrak{y}_1) - \frac{G^2(\mathfrak{y}_2, qm\mathfrak{y}_1)}{A(1,q)} \Upsilon_1 \right| = 0.054$$

and

$$\frac{q(\mathfrak{y}_2 - m\mathfrak{y}_1)G^2(\mathfrak{y}_2, qm\mathfrak{y}_1)}{A(1,q)} \left[\frac{1}{\mathfrak{y}_2(q\mathfrak{y}_2 + (1-q)m\mathfrak{y}_1)}(A_1(q) + A_3(q)) + \frac{1}{m\mathfrak{y}_1^2}(A_2(q) + A_4(q)) \right] = 0.071$$

Thus, $0.054 \leq 0.071$.

Proposition 2 *Let* $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{R}, \mathfrak{y}_1 < \mathfrak{y}_2, \alpha \in 0,1], m \in 0,1]$ *and* $0 < q < 1$. *Then we have*

$$\left| G^2(\mathfrak{y}_2, qm\mathfrak{y}_1) - \frac{A(\mathfrak{y}_2, qm\mathfrak{y}_1)G^2(\mathfrak{y}_2, qm\mathfrak{y}_1)}{A(1,q)} \Upsilon_1 \right|$$

$$\leq \frac{q(\mathfrak{y}_2 - m\mathfrak{y}_1)A(\mathfrak{y}_2, qm\mathfrak{y}_1)G^2(\mathfrak{y}_2, qm\mathfrak{y}_1)}{A(1,q)}$$

$$\times \frac{1}{\mathfrak{y}_2(q\mathfrak{y}_2 + (1-q)m\mathfrak{y}_1)}(A_1(q) + A_3(q)) + \frac{1}{m\mathfrak{y}_1^2}(A_2(q) + A_4(q)) \tag{20}$$

where $\Upsilon_1 = \frac{1}{(\mathfrak{y}_2 - m\mathfrak{y}_1)} \int_{m\mathfrak{y}_1}^{\mathfrak{y}_2} \frac{1}{\mathfrak{r}}\ {}_{m\mathfrak{y}_1}d_q\mathfrak{r} = (1-q)\sum_{n=0}^{\infty} \frac{q^n}{q^n\mathfrak{y}_2 + m(1-q^n)\mathfrak{y}_1}$.

Proof. The inequality (13) for function $\mathfrak{F}(\mathfrak{r}) = \frac{q\mathfrak{y}_2 m\mathfrak{y}_1(qm\mathfrak{y}_1 + \mathfrak{y}_2)}{[2]_q \mathfrak{r}}$ leads to required result. If we take $\mathfrak{y}_1 = 1, \mathfrak{y}_2 = 2, q = 0.5, m = 0.5$ and $\alpha = 0.5$ in (20), we get

$$\left| G^2(\mathfrak{y}_2, qm\mathfrak{y}_1) - \frac{A(\mathfrak{y}_2, qm\mathfrak{y}_1)G^2(\mathfrak{y}_2, qm\mathfrak{y}_1)}{A(1,q)} \Upsilon_1 \right| = 0.188$$

and

$$\frac{q(\mathfrak{y}_2 - m\mathfrak{y}_1)A(\mathfrak{y}_2, qm\mathfrak{y}_1)G^2(\mathfrak{y}_2, qm\mathfrak{y}_1)}{A(1,q)} \left[\frac{1}{\mathfrak{y}_2(q\mathfrak{y}_2 + (1-q)m\mathfrak{y}_1)}(A_1(q) + A_3(q)) + \frac{1}{m\mathfrak{y}_1^2}(A_2(q) + A_4(q)) \right] = 3.27$$

Thus, $0.188 \leq 3.27$.

Conclusion

In the current study, we initially proved two quantum identities using the integration by parts method. Then, using these identities, we established some new Midpoint and Trapezoid type inequalities for differentiable (α, m) -convex functions, which was the main motivation of this paper. In upcoming directions, similar inequalities could be found for co-ordinated convex functions as well.

References

1. Ali A, Gulshan G, Hussain R, Latif A, Muddassar M. Generalized inequalities of the type of Hermite Hadamard-Fejer with Quasi-Convex functions by way of k -Fractional derivative. J Comput Anal Appl. 2017;22:1208-1219. Available from: https://www.researchgate.net/publication/302795285_Generalized_Inequalities_of_the_type_of_Hermite-Hadamard-Fejer_with_Quasi-Convex_Functions_by_way_of_k-Fractional_Derivatives

2. Dragomir SS,. Pearce CEM. Quasi-convex functions and Hadamards inequality. Bull Austral Math Soc. 1998;57:377-385. Available from: https://doi.org/10.1017/S0004972700031786

3. Hussain R, Ali A, Latif A, G Gulshan G. Some k-fractional associates of Hermite-Hadamard's inequality for quasi-convex functions and applications to special means. Fractional Differ Calc. 2017;7:301-309. Available from: http://dx.doi.org/10.7153/fdc-2017-07-13

4. Fink AM. Hadamard's inequality for log-concave functions. Math Comut Model. 2000;32:625-629. Available from: https://www.sciepub.com/reference/173909

5. Dragomir SS. On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane. Taiwan J Math. 2001;5(4):775-788. Available from: https://projecteuclid.org/journals/taiwanese-journal-of-mathematics/volume-5/issue-4/ON-THE-HADAMARDS-INEQULALITY-FOR-CONVEX-FUNCTIONS-ON-THE-CO/10.11650/twjm/1500574995.full

6. Hussain R, Ali A, Latif A, Gulshan G. Co-ordinated convex function of three variables and some analogues inequalities with applications. J Comput Anal and Appl. 2021;29:505-517. Available from: https://www.researchgate.net/publication/342587589_CO-ORDINATED_CONVEX_FUNCTIONS_OF_THREE_VARIABLES_AND_SOME_ANALOGOUS_INEQUALITIES_WITH_APPLICATIONS

Citation: Gulshan G, Ali MA, Budak H, Hussain R. New Quantum Estimates for Midpoint and Trapezoid Type Inequalities Through (α,m)-Convex Functions with Applications. Ann Math Phys. 2024;7(2):222-231. Available from: https://dx.doi.org/10.17352/amp.000126

7. Scan I. Hermite-Hadamard type inequalities for harmonically convex functions. Hacet J Math Stat. 2014; 43:935-942. Available from: https://dergipark.org.tr/tr/download/article-file/711773

8. Niculescu PC. Convexity according to the geometric mean. Math Inequal Appl. 2000;2:155-167. Available from: http://dx.doi.org/10.7153/mia-03-19

9. QiF, Xi BY. Some integral inequalities of Simpson type for GA-convex functions. Georgian Math J. 2013; 20:775-788. Available from: https://www.degruyter.com/document/doi/10.1515/gmj-2013-0043/html

10. Mihesan VG. A generalization of the convexity. In: Seminar on Functional Equations, Approx. and Convex., Cluj-Napoca, Romania. 1993. Available from: https://www.sciepub.com/reference/54446

11. Bubeck S. Convex optimization Algorithms and complexity. Found Trends Mach Learn. 2015;8:231-357. Available from: https://doi.org/10.48550/arXiv.1405.4980

12. Chang TS Jin XX, Luh PB, Miao XIYI. Large-scale convex optimal control problems time decomposition, incentive coordination and parallel algorithm. IEEE T Automat. 1990; 35:108-114. Available from: file:///C:/Users/WEBTEAM/Downloads/LSOptimalControTsu-ShuanChang.pdf

13. Fagbemigun BO, Mogbademu AA. Some classes of convex functions on time scales. RGMIA Research ReportCollections. 2019; 22:1-12. Available from: https://rgmia.org/papers/v22/v22a13.pdf

14. Alp N. Sarikaya MZ. Kunt M, İşcan İ. q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions. J King Saud Univ Sci. 2018;30:193-203. Available from: https://doi.org/10.1016/j.jksus.2016.09.007

15. Ernst T. The History of q-calculus and a New Method, Department of Mathematics, Uppsala University, Sweden, China. 2000. Available from: https://www.scribd.com/document/21401844/The-History-of-Q-calculus-and-a-New

16. Kac V, Cheung P. Quantum Calculus, Springer. 2001.

17. Benatti F, Fannes M, Floreanini R, Petritis D. Quantum Information, Computation and Cryptography: An Introductory Survey of Theory, Technology and Experiments. Springer Science and Business Media. 2010. Available from: https://link.springer.com/book/10.1007/978-3-642-11914-9

18. Bokulich A, Jaeger G. Philosophy of Quantum Information Theory and Entaglement., Cambridge Uniersity Press. 2010.

19. Jackson FH. On a q-definite integrals. Quarterly. J Pure Appl Math. 1910;4:193-203.

20. Al-Salam W. Some fractional q -integrals and q -derivatives, Proc. Edinburgh Math Soc. 1966; 15(2):135-140.

21. Tariboon J, Ntouyas SK. Quantum calculus on finite intervals and applications to impulsive difference equations. Adv Differ Equ. 2013;1-19. Available from: https://advancesincontinuousanddiscretemodels.springeropen.com/articles/10.1186/1687-1847-2013-282

22. Bermudo S, Kórus P, Valdés JN. On q-Hermite-Hadamard inequalities for general convex functions. Acta Math Hung. 2020;162:364-374.

23. Ali MA, Budak H, Abbas M, Chu YM. Quantum Hermite-Hadamard-type inequalities for functions with convex absolute values of second q^{η_2} -derivatives. Adv Differ Equ. 2021; 2021:1-12. Available from: https://advancesincontinuousanddiscretemodels.springeropen.com/articles/10.1186/s13662-020-03163-1

24. Ali MA, Alp N, Budak H, Chu YM, Zhang Z. On some new quantum midpoint type inequalities for twice quantum differentiable convex functions. Open Math. 2021;9:427-439. Available from: http://dx.doi.org/10.1515/math-2021-0015

25. Alp N, Sarikaya MZ. Hermite Hadamard's type inequalities for co-ordinated convex functions on quantum integral. Appl. Math. E-Notes. 2020; 20:341-356. Available from: https://www.emis.de/journals/AMEN/2020/AMEN-190629.pdf

26. Budak H. Some trapezoid and midpoint type inequalities for newly defined quantum integrals. Proyecciones. 2021; 40:199-215. Available from: https://doi.org/10.22199/issn.0717-6279-2021-01-0013

27. Budak H, Ali MA, Tarhanaci M. Some new quantum Hermite-Hadamard-like inequalities for coordinated convex functions. J Optim Theory Appl. 2020;186:899-910. Available from: https://link.springer.com/article/10.1007/s10957-020-01726-6

28. Jhanthanam S, Tariboon J, Ntouyas SK, Nonlaopon K. On q -Hermite-Hadamard inequalities for differentiable convex functions. Mathematics. 2019;7:632.

29. Liu W, Hefeng Z. Some quantum estimates of Hermite-Hadamard inequalities for convex functions. J Appl Anal Comput. 2016; 7:501-522. Available from: https://www.mdpi.com/2227-7390/7/2/152#

30. Noor MA, Noor KI, Awan MU. Some quantum estimates for Hermite-Hadamard inequalities. Appl Math Comput. 2015; 251:675-679. Available from: https://www.sciencedirect.com/science/article/abs/pii/S0096300314016221

31. Noor MA, Noor KI, Awan MU. Some quantum integral inequalities via preinvex functions. Appl Math Comput. 2015; 269:242–251. Available from: https://doi.org/10.1016/j.amc.2015.07.078

32. Nwaeze ER, Tameru AM. New parameterized quantum integral inequalities via η -quasiconvexity. Adv Differ Equ. 2019;2019:1-12. Available from: https://link.springer.com/article/10.1186/s13662-019-2358-z

33. Khan MA, Noor M, Nwaeze ER, Chu YM. Quantum Hermite–Hadamard inequality by means of a Green function. Adv Differ Equ. 2020;2020:1-20. Available from: https://advancesincontinuousanddiscretemodels.springeropen.com/articles/10.1186/s13662-020-02559-3

34. Budak H, Erden S, Ali MA. Simpson and Newton type inequalities for convex functions via newly defined quantum integrals. Math Meth Appl Sci. 2020; 44:378-390. Available from: https://doi.org/10.1002/mma.6742

35. Ali MA, Budak H, Zhang Z, Yildrim H. Some new Simpson's type inequalities for co-ordinated convex functions in quantum calculus. Math Meth Appl Sci. 2021; 44:4515-4540. Available from: https://doi.org/10.1002/mma.7048

36. Ali MA, Abbas M, Budak H, Agarwal P, Murtaza G, Chu YM. New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions. Adv Differ Equ. 2021;2021:1-21. Available from: https://advancesincontinuousanddiscretemodels.springeropen.com/articles/10.1186/s13662-021-03226-x

37. Vivas-Cortez M, Ali MA, Kashuri A, Sial IB, Zhang Z. Some new Newton's type integral inequalities for coordinated convex functions in quantum calculus. Symmetry. 2020;12:1476. Available from: https://www.mdpi.com/2073-8994/12/9/1476

38. Ali MA, Chu YM, Budak H, Akkurt A, Yildrim H. Quantum variant of Montgomery identity and Ostrowski-type inequalities for the mappings of two variables. Adv Differ Equ. 2021; 2021:1-26. Available from: https://advancesincontinuousanddiscretemodels.springeropen.com/articles/10.1186/s13662-020-03195-7

39. Ali MA, Budak H, Akkurt A, Chu YM. Quantum Ostrowski type inequalities for twice quantum differentiable functions in quantum calculus. Open Math. 2021;19:427-439. Available from: http://dx.doi.org/10.1515/math-2021-0020

40. Budak H, Ali MA, Alp N, Chu YM. Quantum Ostrowski type integral inequalities. J Math Inequal. 2021.

Citation: Gulshan G, Ali MA, Budak H, Hussain R. New Quantum Estimates for Midpoint and Trapezoid Type Inequalities Through (α,m)-Convex Functions with Applications. Ann Math Phys. 2024;7(2):222-231. Available from: https://dx.doi.org/10.17352/amp.000126

41. Sudsutad W, Ntouyas SK, Tariboon J. Quantum integral inequalities for convex functions. Journal of Mathematical Inequalities. 2015;9:781-793. Available from: https://www.researchgate.net/profile/Weerawat-Sudsutad-2/publication/281759795_Quantum_integral_inequalities_for_convex_functions/links/56e8497408aea51e7f3b39eb/Quantum-integral-inequalities-for-convex-functions.pdf

42. Kirmaci S. Inequalities for differentiable functions and applications to special means of real numbers and to midpoint formula. Appl Math Comput. 2014;147:137-146.

43. Dragomir SS, Agarwal RP. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Appl Math Lett. 1998;11:91-95. Available from: https://doi.org/10.1016/S0893-9659(98)00086-X

44. Dragomir SS, Pearce CEM. Selected Topics on Hermite-Hadamard Inequalities and Applications., RGMIA Monographs, Victoria University. 2000. Available from: https://rgmia.org/papers/monographs/Master.pdf

45. Ali MA, Budak H, Nanlaopon K, Abdullah Z. Simpson's and Newton's Type Inequalities for (η_1, m) -convex functions via Quantum Calculus. Submitted to AIMS Mathematics.

46. Pearce CEM, Peccaric J. Inequalities for differentiable functions with application to special means and quadrature formula. Appl Math Lett. 2000;13: 51-55. Available from: https://doi.org/10.1016/S0893-9659(99)00164-0

47. Rovelli C. Quantum Gravity (Cambridge Monograph On Math. Physics. 2004. Available from: https://doi.org/10.1017/CBO9780511755804

48. Sengar RS, Sharmssssa M, Trivedi A. International Journal of Civil Engineering and Technology. 2015;6:34–44.

Citation: Gulshan G, Ali MA, Budak H, Hussain R. New Quantum Estimates for Midpoint and Trapezoid Type Inequalities Through (α,m)-Convex Functions with Applications. Ann Math Phys. 2024;7(2):222-231. Available from: https://dx.doi.org/10.17352/amp.000126