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Research Article

Analytical Solution of the Steady Navier-Stokes Equation for an Incompressible Fluid Entrained by a Rotating Disk of Finite Radius in the Area of Boundary Layer

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Abstract

The flow in the neighborhood of a rotating disk is of great practical importance, particularly in connection with rotary machines. It becomes turbulent at larger Reynolds numbers, $R > 3 \cdot 10^5$, in the same way as the flow about a plate. In this article, we consider a motion of incompressible fluid that is always turbulent in azimuthal direction (Reynolds number based on azimuthal velocity $R_\varphi > 3 \cdot 10^5$) and is of both kinds in a radial direction, i.e. laminar (Reynolds number based on radial velocity $R_r < 3 \cdot 10^5$) and turbulent ($R_r > 3 \cdot 10^5$). The equations of analyticity of functions of a spatial complex variable (shortly, the equations of tunnel mathematics) afford a possibility to seek the solutions of steady Navier-Stokes equation in view of elementary functions. All vector fields, including those obeying the Navier-Stokes equation, satisfy the equations of tunnel mathematics. The Navier-Stokes equations themselves are afterward applied for verification of obtained solutions and calculation of the pressure. Obtained formulae for pressure allow us to visualize the presence of the boundary layer and estimate its thickness for laminar and turbulent flows. We use Prandtl`s concept of considering fluids with small viscosities, i. e. we suppose that the Reynolds number is enough large and the viscosity has an important effect on the motion of fluid only in a very small region near the disk (boundary layer). We also suppose that the fluid and the disk had at the beginning the same temperatures and the energy dissipation occurs only by means of internal friction.

Introduction

The exact solution of the problem indicated in the title of this article for an infinite disk (the problem was stated by Th. von Karman) is given in the handbook on theoretical physics of Landau and Lifshitz [1] and in the handbook of Schlichting [2]. There this problem was reduced to solving a system of second-order ordinary differential equations which was obtained by numerical methods. The results of this solution are shown in Figure 1.

In Figure 1 the function F corresponds to the radial velocity v_r, the function G to the azimuthal velocity v_φ, and the function H to the axial projection v_z.

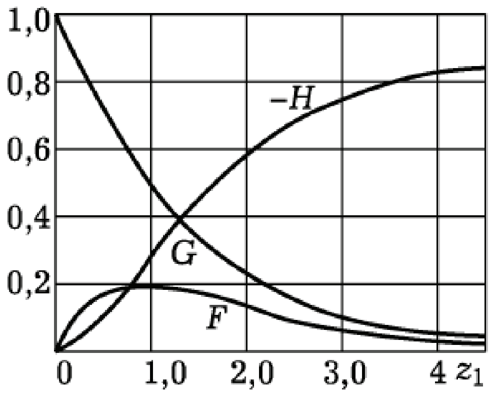

Figure 1: Solution of the problem for an infinite disk using numerical methods in the handbook of Landau and Lifshitz.

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In this case, the disk is located in the plane $z=0$ of cylindrical coordinates and rotates around the z-axis with a constant angular velocity Ω. The liquid is considered from the side of the disk where $z>0$. The boundary conditions for the problem are as follows:

in the plane $z=0$ we have $v_r=0; v_\varphi=\Omega r; v_z=0$, (1)

i. e. the fluid adheres to the surface of the disk. We assumed in our problems that the radius of the disk is arbitrarily equal to unity. Although we have imposed no boundary conditions at $r=\infty$, and $z=H$, where H is an immersion depth of the disk, the equations of tunnel mathematics allow us to calculate the correct solution of the problem at some distance from the edge of the disc (for r approximately equal 2).

This article provides a solution to a more realistic physical problem when a rotating disk has a finite radius (arbitrarily equal to unity). The solution applies only to the volume of the fluid enclosed in a cylinder of approximately unit radius of height z_1 (where z_1 is the vicinity of z coordinate on which the Navier-Stokes equations work (34) and (34a). In the limits of z_1 the fluid moves in the horizontal, mainly radial direction on Figure 2).

As is known, if the nonlinear terms in the Navier-Stokes equations for a viscous fluid do not vanish identically then the solution of these equations presents great difficulties, and exact solutions can be obtained only in a very small number of cases. Concerning the unsteady Navier-Stokes equation, T. Tao in his article [3] showed that it can have solutions that become infinite during finite time (blowup solutions). And if even closed-form solutions exist they are usually presented in the form of infinite power series which is inconvenient for use in practice. That`s why it is reasonable to seek solutions of steady Navier-Stokes equation in the form of elementary functions. An exact solution to the problem under consideration (purely mathematical and also in the form of a power series) was given by Miss D. M. Hannah [4], and A.N. Tifford [5] for the case of laminar flow; H. Schlichting and E. Truckenbrodt [6] provided an approximate solution with the aid of numerical methods. E. Truckenbrodt also investigated the case of turbulent flow. D. Weyburne [7] applied this to describe the boundary layer in the standard probability distribution function methodology. Currently used analytical formulae for velocity components in the turbulent boundary layer over rotating disk can be found in [8].

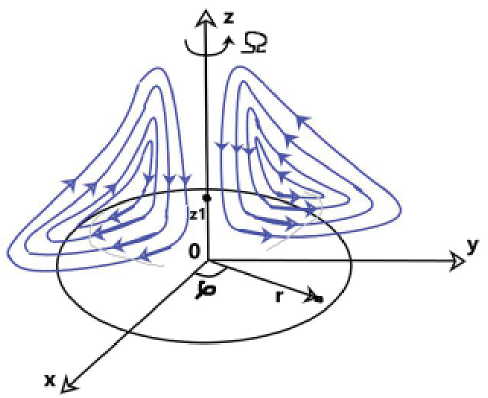

Figure 2: Fluid streamlines in one of the planes parallel to the z-axis.

Theory

Just as for an infinite disk, the solution of the problem for a fluid entrained by a rotating disk of finite radius is symmetric with respect to the φ coordinate (i.e. has axial symmetry). Figure 2 schematically shows fluid streamlines in one of the planes parallel to the z-axis.

Tunnel mathematics equations applied to the components of the vector velocity field in the Cartesian coordinate system look like this [9]:

$$\frac{\partial u}{\partial x}+\frac{wxy}{(x^2+y^2)^{3/2}}=\frac{\partial v}{\partial y}-\frac{wxy}{(x^2+y^2)^{3/2}}=-\frac{2y}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial z}+\frac{\partial w}{\partial z}. \quad (2)$$

$$\frac{\partial v}{\partial x}+\frac{wy^2}{(x^2+y^2)^{3/2}}=-\frac{\partial u}{\partial y}+\frac{wx^2}{(x^2+y^2)^{3/2}}=-\frac{2y}{\sqrt{x^2+y^2}}\frac{\partial v}{\partial z}. \quad (3)$$

$$\frac{\partial w}{\partial x}=-i\frac{\partial w}{\partial y}=-\frac{\partial u}{\partial z}-i\frac{\partial v}{\partial z}. \quad (4)$$

The components of the velocity field in expressions (2) - (4) are set as follows:

$$V_x=u; V_y=v; V_z=Rew; \quad (5)$$

where Re denotes the real part of the function; moreover, the functions u, v, w are components of the function of the spatial complex variable [9]:

$$P(L)=u(x,y,z)+iv(x,y,z)+fw(x,y,z); \quad (6)$$

As usual, the fluid is considered as incompressible:

$$\nabla\cdot V=div\boldsymbol{V}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \quad (7)$$

Taking into account (7) we obtain from equations (2) such relation:

$$\frac{\partial u}{\partial z}=0. \quad (8)$$

Relation (8) holds provided

$$\frac{\partial w}{\partial z}=0. \quad (9)$$

We recall that function w is an analytic function of the variables x and y and, therefore, a harmonic function on the xy plane.

Any vector fields including those obeying the Navier-Stokes equations satisfy the relations (2) - (4).

Taking into account the symmetry in the φ coordinate the Navier-Stokes equations in cylindrical polar coordinates take the following form:

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$$\rho\left(v_r\frac{\partial v_r}{\partial r}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\varphi^2}{r}\right)=-\frac{\partial P}{\partial r}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)+\frac{\partial^2 v_r}{\partial z^2}-\frac{v_r}{r^2}\right); \quad (10)$$

$$\rho\left(v_r\frac{\partial v_\varphi}{\partial r}+v_z\frac{\partial v_\varphi}{\partial z}+\frac{v_r v_\varphi}{r}\right)=\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\varphi}{\partial r}\right)+\frac{\partial^2 v_\varphi}{\partial z^2}-\frac{v_\varphi}{r^2}\right); \quad (11)$$

$$\rho\left(v_r\frac{\partial v_z}{\partial r}+v_z\frac{\partial v_z}{\partial z}\right)=-\rho g-\frac{\partial P}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{\partial^2 v_z}{\partial z^2}\right); \quad (12)$$

where g is the gravitational acceleration;

P – pressure in fluid;

μ - coefficient of dynamic viscosity;

ρ - fluid density.

Due to the fact that the Laplacian of the analytic function w is equal to zero, the equations of tunnel mathematics describe a fluid that moves along the z-axis without internal friction, i.e. as an ideal fluid (this is a feature of this method for incompressible fluid). Taking this into account we can obtain the following form of the function w from equations (5), (9), and (12):

$$w=log re^{i\varphi}=log r+i\varphi=log\sqrt{x^2+y^2}+i\tan^{-1}\frac{y}{x}. \quad (13)$$

In order to comply with dimension, the expression (13) must be written in the following form:

$$w=\Omega C_0\left(\log\frac{r}{r_0}+i\varphi\right)=\Omega C_0\left(\log\frac{\sqrt{x^2+y^2}}{r_0}+i\tan^{-1}\frac{y}{x}\right), \quad (14)$$

Where the constants have follow dimensions: $[C_o] = m$, $[r_o] = m$; we can impose $r_o = 1\ m$. Constant r_o allows to adjustment of the arbitrary radius of the disk.

We find such relations after differentiating (14):

$$\frac{\partial w}{\partial x}=\frac{\Omega C_0(x-iy)}{x^2+y^2}; \quad (15)$$

$$\frac{\partial w}{\partial y}=\frac{\partial v}{\partial z}=\frac{\Omega C_0(y+ix)}{x^2+y^2}=\frac{\Omega C_0 f^*}{\sqrt{x^2+y^2}}=\frac{\Omega C_0\left(f+\frac{2y}{\sqrt{x^2+y^2}}\right)}{\sqrt{x^2+y^2}}=\frac{\Omega C_0\left(f+2\sin\varphi\right)}{r}. \quad (16)$$

Expression (16) uses the following relations of tunnel mathematics for the operator f and its conjugate operator f^* [9]:

$$f=ie^{i\varphi}=-\sin\varphi+i\cos\varphi=\frac{-y+ix}{\sqrt{x^2+y^2}}. \quad (17)$$

$$f^*=ie^{-i\varphi}=\sin\varphi+i\cos\varphi=\frac{y+ix}{\sqrt{x^2+y^2}}. \quad (18)$$

$$f\times f^*=-1. \quad (19)$$

$$f^*=f+2\sin\varphi. \quad (20)$$

Special attention should be paid to the operator f in (16) since the factor at it will be used to reconstruct the spatial coordinate z in the equations on the plane (what corresponds to the relation (6)). We can see from equation (16) that this factor is r (we omit constants and terms depending on φ).

Generally speaking, the complex function w must be selected in such a way that its real part corresponds to the pattern of occurring physical phenomena (i. e. to the projection V_z of the velocity field in Figure 2), and the integrating of relations (2) and (3) does not generate integrals that could not be solved in quadratures.

The form of the real part of the function w according to expression (14) is shown in Figure 3.

It can be seen from the graph shown in Figure 3 that the radius of the rotating disk should be equal to unity (arbitrarily). For $r>1$ the fluid streamlines are already rising.

This form of the real part of the function w is inapplicable in the vicinity of the point $r=0$ since the graph goes to infinity.

Transforming relations (2) and (3) to cylindrical polar coordinates and taking into account expressions (8) and (9) we obtain:

$$\frac{\partial v_r}{\partial r}-\frac{\partial v_\varphi}{\partial r}\tan\varphi+\frac{v_r}{r}+\frac{v_\varphi}{r}\cot\varphi+\frac{w\sin\varphi\cos\varphi}{r}$$
$$=\frac{\partial v_r}{\partial r}+\frac{\partial v_\varphi}{\partial r}\cot\varphi+\frac{v_r}{r}-\frac{v_\varphi}{r}\tan\varphi-\frac{w\sin\varphi\cos\varphi}{r}=0; \quad (21)$$

$$\frac{\partial v_r}{\partial r}\tan\varphi+\frac{\partial v_\varphi}{\partial r}-\frac{v_r}{r}\cot\varphi+\frac{v_\varphi}{r}+\frac{w\sin^2\varphi}{r}$$
$$=-\frac{\partial v_r}{\partial r}\cot\varphi+\frac{\partial v_\varphi}{\partial r}+\frac{v_r}{r}\tan\varphi+\frac{v_\varphi}{r}+\frac{w\cos^2\varphi}{r}=-\frac{(2\sin\varphi)\Omega C_0\left(f+2\sin\varphi\right)}{r}. \quad (22)$$

When deriving equations (21) and (22) the well-known formulas were used for the transition from Cartesian velocity projections to cylindrical polar ones:

$$u=v_r\cos\varphi-v_\varphi\sin\varphi; \quad (23)$$

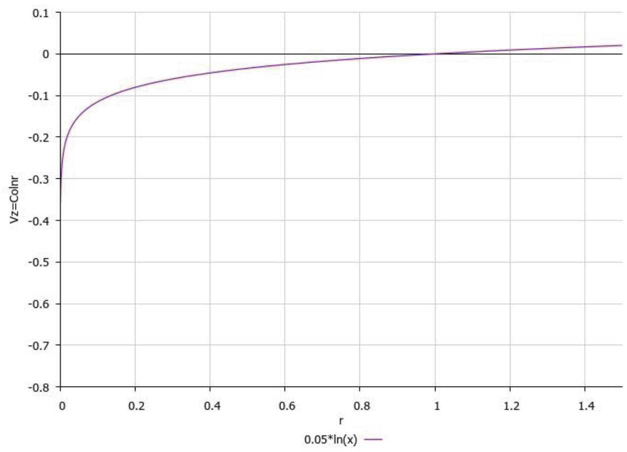

Figure 3: The graph of the real part of the function w ($C_0 = 0.05\ m$).

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$$v = v_r \sin\varphi + v_\varphi \cos\varphi. \quad (24)$$

Since in the region under consideration (Figure 2) the liquid moves horizontally (mainly radially), when solving equations (21) and (22) we can put $w = 0$. By integrating these equations under such conditions we obtain two solutions for each of the projections of the velocity v_r и v_φ. The solutions that meet the physical pattern of the phenomena of this problem (particularly the boundary conditions (1)) are as follows:

$$v_\varphi = \Omega r; \quad (25)$$

$$v_r = \frac{\Omega\sigma}{r}, \quad (26)$$

where σ is a constant of integration.

Expressions (25) and (26) are the solutions of equations (21).

Expression (26) holds since the solution should not have a dependence on φ, and we work with the assumption (9). Expression (26) is true for $r > 1$, however, it doesn`t work for $r \to 0$ (it tends to infinity). That`s why it needs a correction.

Results and discussions

To extend solutions (25) and (26) into space we use the factor at the operator *f* in equation (22):

$$z = \frac{1}{r}. \quad (27)$$

We consider *sinφ* at (22) as a constant.

The specific form of the velocity projection v_r depends on the integral relation representing the corresponding mass fluxes of fluid through the surfaces of the cylinder covering the region in question (Figure 2).

Extension of solution on a plane into space for radial velocity

So, for the projection v_r the following integral relation is used:

$$\int v_z r dr = \int v_r dz, \quad (28)$$

where the integral on the left-hand side represents the mass flux entering through the upper end of the cylinder and the integral on the right-hand side represents the mass flux that is scattered through the side surface of the cylinder.

Taking into account equalities (5), (14), and (26), we can rewrite the integral relation (28) in the following manner:

$$\int r \log r dr = \int \frac{1}{r} dz, \quad (29)$$

where constants are omitted.

Accordingly (27), the presence of the factor $\frac{1}{r}$ in the right-hand side of (29) is sufficient for building of spatial formula. In order for the integral in the right-hand side of (29) to not depend on changing the variable of integration *z* by *r*, both variables *r* and *z* should enter in the expression for radial velocity v_r symmetrically. Then formula (26) can be extended into space as follows:

$$v_r = -\Omega \frac{rz}{z_{0r}} \log \frac{rz}{r_0 z_{0r}}; \quad (30)$$

where the range of *r* is about (0; 2); *z* varies in the limits of z_r.

In (30) z_{0r} is a normalization factor, and its dimension is a meter. The value of z_{0r} depends on the value of the *z* coordinate at which the Navier-Stokes equation works. Particularly, for the *z* coordinate corresponding to the boundary layer the value of z_{0r} is of the order of the thickness of boundary layer δ, i. e. $z_{0r} = \delta$.

Approximate spatial and plane graphs for the functions v_r in (30) and corrected (26) are shown in Figures 4,5 respectively. These graphs correspond to the pattern of physical phenomena arising during the rotation of a disk in a liquid for the following

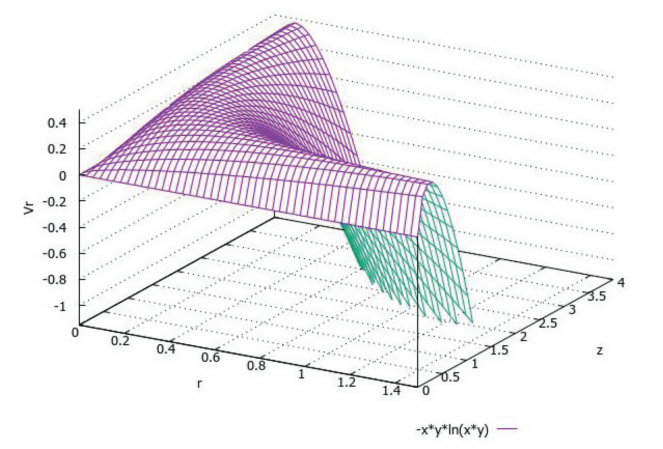

Figure 4: Spatial graph for formula (30).

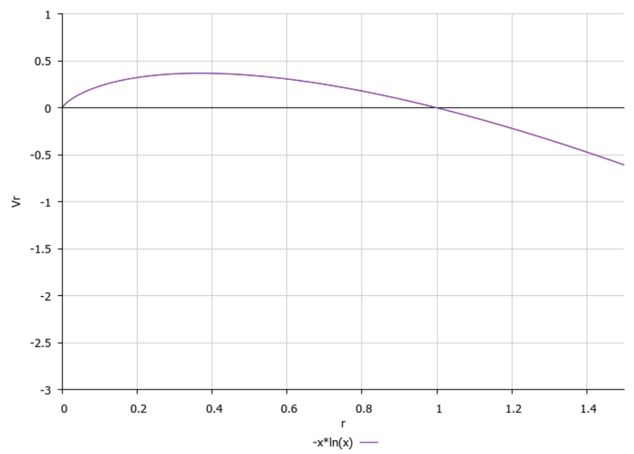

Figure 5: Plane graph for corrected formula (26).

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reasons. Let us consider from above the motion of a liquid over a disk at rest in a non-inertial frame of reference for a small value of z coordinate, when the cohesion forces of liquid particles with the disk surface still have an appreciable value (Figure 6).

Coriolis forces $\vec{F}_{k\varphi}$ caused by the azimuthal velocity $\vec{v}_\varphi^{\,'}$ (the prime means movement in a non-inertial frame of reference) will be directed to the origin and thus will prevent the increase in radial velocity $\vec{v}_r^{\,'}$ under the action of centrifugal forces $\vec{F}_{cf}$. At the same time the Coriolis forces $\vec{F}_{kr}$ caused by the radial velocity $\vec{v}_r^{\,'}$ will contribute to an increase in the magnitude of the azimuthal velocity $\vec{v}_\varphi^{\,'}$. As a result of such a dynamic pattern the radial velocity $\vec{v}_r^{\,'}$ having reached its maximum will gradually decrease to zero which corresponds to the graphs shown in Figures 4,5.

A similar picture for the same reasons will be observed as the z coordinate increases while the r coordinate remains constant. In this case the radial velocity $\vec{v}_r^{\,'}$ begins to increase due to the weakening of the cohesion forces between the liquid particles and the disk surface.

This dependence of the radial velocity component v_r on the coordinates r and z can also be explained by energy considerations. If we return to the inertial reference system, then the appearance of a new component of velocity (radial) in it requires certain energy expenditures. Since the energy supplied to the disk for its rotation remains unchanged, an increase in the radial component of the velocity v_r is possible only due to a decrease in the azimuthal component v_φ which in the inertial frame of reference will tend to retain its value. Therefore, the process of extinguishing the radial component v_r in both directions, r, and z, will inevitably occur in the liquid.

In this regard, a natural question arises as to why in the handbook of Landau and Lifshitz [1] and in the handbook of Schlichting [2] the radial velocity depends on the coordinate r according to linear law:

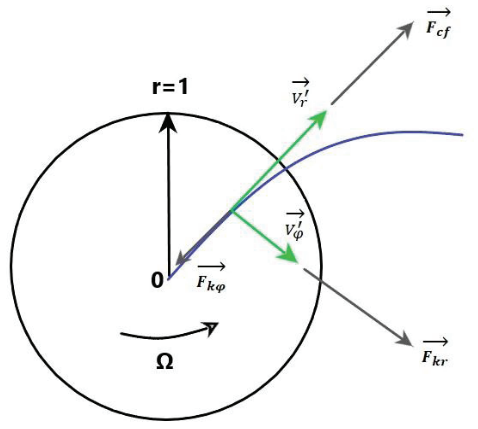

Figure 6: Motion of liquid over a disk at rest in a non-inertial frame of reference for a small value of *z* coordinate.

$$v_r = r\Omega F(z). \tag{31}$$

The point is that formula (31) really takes place for small values of the coordinate r. And since for an infinite disk (namely such a model is considered in the handbook of Landau and Lifshitz and in the handbook of Schlichting) any value of r in principle can be considered as small, then formula (31) has a full right to live. It is of course inapplicable for a disk of finite radius.

Since in the region under consideration in Figure 2 the liquid moves horizontally (mainly radially), then the projection of the particle velocity on the z-axis is zero ($v_z = 0$), and therefore Liouville's theorem on the invariability of the phase volume of a system of particles is satisfied automatically [10]. The fivefold integral $\int dr dz dv_r dv_\varphi dv_z$ is identically equal to zero.

It should also be noted that in accordance with the graphs in Figures 4,5 for $r > 1$ the fluid streamlines must abruptly change the direction of motion so that the vector of the total velocity makes an obtuse angle with the r axis as actually shown in Figure 2.

Extension of solution on a plane into space for azimuthal velocity

By means (27) the extension of (25) into space is fulfilled very simple.

$$v_\varphi = \frac{\Omega r}{\dfrac{z}{z_{0\varphi}} + 1}; \tag{32}$$

where $z_{0\varphi}$ is a normalization factor, and its dimension is a meter. We added unity in the denominator in order to satisfy the boundary conditions (1).

Finding the auxiliary relations

We can at once verify the solutions (30) and (32) that have been expanded into space by means apparatus of tunnel mathematics with the aid of true Navier-Stokes equations. The simplest way to do this is to use equation (11) since it doesn't contain the pressure P. Substituting solutions (30) and (32), and the real part of relation (14), and their corresponding derivative into equation (11) we conclude following important auxiliary relation:

$$\rho\left(\frac{2v_r}{r} - \frac{v_z}{z + z_{0\varphi}}\right) = \frac{2\mu}{(z + z_{0\varphi})^2}. \tag{33}$$

In the area of laminar boundary layer $z_{0\varphi} \gg z = \delta$, so we have such a quadratic equation for determining of $z_{0\varphi}$:

$$z_{0\varphi}^2\left(\frac{2\rho v_r}{r}\right) + z_{0\varphi}(-\rho v_z) - 2\mu = 0\,. \tag{34}$$

In the area of turbulent boundary layer $z_{0\varphi} \ll z = \delta$, so we have a similar quadratic equation for determining of δ:

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$$\delta^2\left(\frac{2\rho v_r}{r}\right)+\delta\left(-\rho v_z\right)-2\mu=0\,. \tag{34a}$$

Formula (11) where the pressure is absent due to axial symmetry can be used to calculate the friction force acting in a fluid including that acting per unit disk surface in a direction perpendicular to its radius. Having performed the appropriate calculations with (32), we obtain the following formula for the friction force:

$$F_{fr}=\mu\frac{\partial v_\varphi}{\partial z}=-\frac{\mu\Omega r}{z_{0\varphi}\left(\dfrac{z}{z_{0\varphi}}+1\right)^2}. \tag{35}$$

Directing z coordinate to zero in expression (35) we obtain the relation for the friction force acting on a unit surface of the disk in the direction perpendicular to its radius:

$$F_{fr}=-\frac{\mu\Omega r}{z_{0\varphi}}. \tag{36}$$

Thus we conclude from (36) that constant $z_{0\varphi}$ for laminar and turbulent motion of fluid must be determined experimentally.

Calculation of pressure and parameters of the boundary layer using the Navier-Stokes equations

Using this method, it is still quite difficult to obtain an exact analytical expression for pressure since, for example, equation (12) for projections of velocity on the z-axis works only for an ideal fluid (i. e. this equation doesn`t work fully in the region of the boundary layer).

The approximate formula for the boundary layer in the area under consideration in Figure 2 looks like this:

$$P=-\frac{\rho(\Omega r)^2}{2}\left(\left(\frac{z}{\delta}log\frac{rz}{r_0\delta}\right)^2-\frac{C_0}{\delta}\left(log^2\frac{r}{r_0}+\left(log\frac{r}{r_0}-\frac{1}{2}\right)log\frac{z}{\delta}\right)-\frac{1}{\left(\dfrac{z}{z_{0\varphi}}+1\right)^2}\right)$$
$$-\frac{\mu\Omega}{2\delta}\left(4z\,log\frac{rz}{r_0\delta}+\frac{r^2}{z}\right). \tag{37}$$

Formula (37) is obtained by integrating equation (10). Since (37) referred to the region of the boundary layer we replaced the constant z_{0r} by the thickness of the boundary layer δ.

In this case, the corresponding derivatives were calculated using formulae (14), (30) and (32). Generally speaking, we must yet add to the right-hand side of (37) the hydrostatic pressure $\rho g\left(H-z\right)$, where H is the immersion depth of the disk.

In Figures 7,8 graphs corresponding to formula (37) in the region of the laminar boundary layer (Reynolds number based on radial velocity $R_r=\frac{v_r r}{\nu}\approx\frac{\Omega z r}{\nu}<3\cdot10^5$, where ν is a kinematic viscosity of fluid) are presented at values of angular velocity Ω

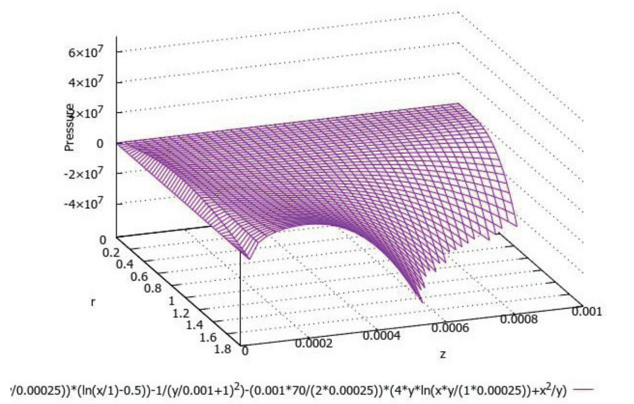

Figure 7: Graph corresponding to formula (37) at *Ω = 70 rad/s* (laminar motion in the radial direction).

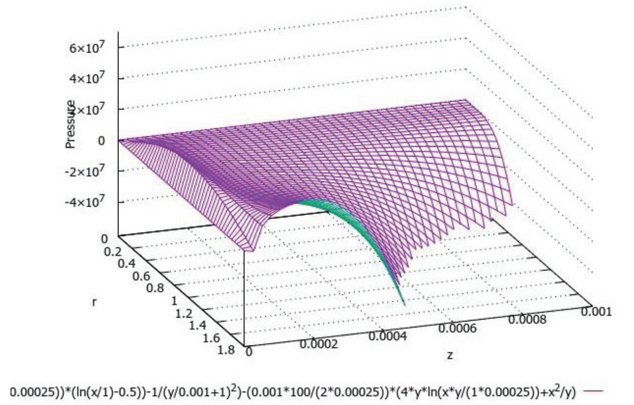

Figure 8: Graph corresponding to formula (37) at *Ω = 100 rad/s* (laminar motion in the radial direction).

equal to *70 rad/s* and *100 rad/s* respectively using the following values of the constants:

$$z_{0\varphi}=0.001m;$$
$$\delta=z_{0r}=0.00025m; \tag{38}$$
$$C_0=0.001m;$$
$$r_0=1m.$$

In all figures shown below the pressure is measured in Pascals, and r and z coordinates are measured in meters.

We recall that the radius of the disc is equal to *1* meter.

The graphs are calculated for water with a density $\rho = 1000\ kg/m^3$ and a dynamic viscosity coefficient $\mu = 0.001\ Pa\times s$. We can see from these graphs that the pressure profile in the boundary layer of incompressible fluid for our problem is convex. In the thin boundary layer (very small z) the pressure significantly arises which corresponds to the presence of a large tangential-velocity gradient, resulting in a large viscous dissipation of energy [1].

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We conclude from (38) that the thickness of the laminar boundary layer is of the order

$\delta \sim 2.5\cdot 10^{-4}\,m$, what fully corresponds to the theory of incompressible fluid [1,2]:

$$\delta \sim \sqrt{\frac{\nu}{\Omega}} = \sqrt{\frac{10^{-6}\,m^2/s}{1001/s}} = 10^{-4}\,m \tag{39}$$

In Figure 9 graph corresponding to formula (37) in the region of the turbulent boundary layer (Reynolds number based on radial velocity $R_r = \frac{\nu_r r}{\nu} \approx \frac{\Omega z r}{\nu} \sim 10^8$) are presented at the value of angular velocity Ω equal to *10000 rad/s* using the following values of the constants:

$$z_{0\varphi} = 0.0001\,m;$$

$$\delta = z_{0r} = 0.025\,m; \tag{40}$$

$$C_0 = 0.001\,m;$$

$$r_0 = 1\,m.$$

We conclude from (40) that the thickness of the turbulent boundary layer is of the order

$\delta \sim 2.5\cdot 10^{-2}\,m$, what fully corresponds to the theory of incompressible fluid [2]:

$$\delta \sim r^{3/5}\left(\frac{\nu}{\Omega}\right)^{1/5} = r^{3/5}\left(\frac{10^{-6}\,m^2/s}{100001/s}\right)^{1/5} = r^{3/5}\cdot 10^{-2}\,m\,. \tag{41}$$

Now we verify either one can find the relations corresponding to (39) and (41) of the theory of boundary layer from obtained formulae for pressure.

Integrating (12) we obtain the following expression for pressure in the region of the boundary layer:

$$P = -\frac{\rho(\Omega C_0)^2}{2\delta}\left(-\frac{z^2}{C_0}\left(log\frac{rz}{r_0\delta}-\frac{1}{2}\right)+\delta log^2\frac{r}{r_0}+\frac{2\delta}{(\Omega C_0)^2}gz\right). \tag{42}$$

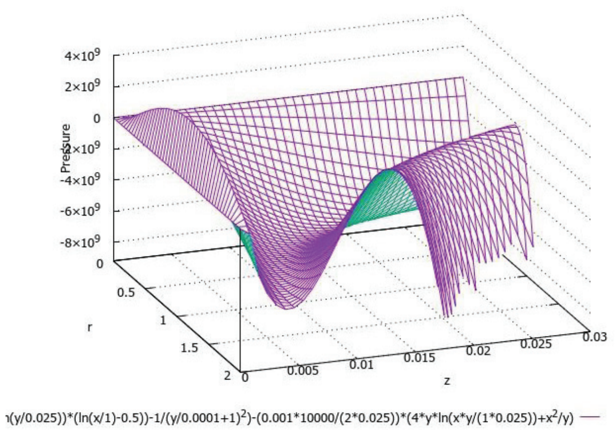

Figure 9: Graph corresponding to formula (37) at *Ω = 10000 rad/s* (turbulent motion in the radial direction).

Corresponding derivatives were calculated using formulae (14) and (30). We omit the constants of integration. In (42) the term with μ is absent.

Comparing (37) and (42) we find:

$$r^2\left(\left(\frac{z}{\delta}log\frac{rz}{r_0\delta}\right)^2-\frac{C_0}{\delta}\left(log^2\frac{r}{r_0}+\left(log\frac{r}{r_0}-\frac{1}{2}\right)log\frac{z}{\delta}\right)-\frac{1}{\left(\frac{z}{z_{0\varphi}}+1\right)^2}\right)$$

$$=-\frac{z^2C_0}{\delta}\left(log\frac{rz}{r_0\delta}-\frac{1}{2}\right)+C_0^2log^2\frac{r}{r_0}-\frac{2gH}{\Omega^2}, \tag{43}$$

where H is the immersion depth of the disk.

We can neglect in (43) the terms with C_0 as we can arbitrarily take its value very small. Besides, we consider polar coordinate r as a constant.

Such ratios in the region of the boundary layer follow from values of constants in (38) and (40):

$$\frac{z}{\delta} \sim 1; \tag{44}$$

for laminar layer: $\frac{z}{z_{0\varphi}} \sim 0$; (45)

for turbulent layer: $\frac{z}{z_{0\varphi}} \gg 1$. (46)

Expanding the logarithms in (43) in Taylor's series near 1 as far as linear terms and neglecting infinitesimal terms we obtain the following approximate equalities.

For laminar layer:

$$r^2\left(\left(\frac{z}{\delta}\left(\frac{z}{\delta}-1\right)\right)^2-1\right) = -\frac{2gH}{\Omega^2}. \tag{47}$$

It`s easy to see from (47) that the maximum power of the z coordinate is 4; therefore, the following relation at the same r coordinate holds:

$$z \sim \frac{const}{\sqrt{\Omega}}. \tag{48}$$

It follows from (48) that the thickness of the laminar boundary layer decreases inversely proportionally to the square root of Ω. This fully corresponds to (39) and the theory of laminar boundary layer in incompressible fluid [1,2].

For turbulent layer:

$$r^2\left(\left(\frac{z}{\delta}\left(\frac{z}{\delta}-1\right)\right)^2-\left(\frac{z_{0\varphi}}{z}\right)^2\right) = -\frac{2gH}{\Omega^2}. \tag{49}$$

We can rewrite (49) in such matter:

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$$\left(\frac{z^2}{\delta z_{0\varphi}}\left(\frac{z}{\delta}-1\right)\right)^2+2gH\left(\frac{z}{\Omega r z_{0\varphi}}\right)^2-1=0. \qquad (50)$$

To find exact solutions of (50) is difficult enough. But we know accordingly (41) that the z coordinate in the region of the turbulent boundary layer must be of the order of *0.01 m*. So, we must determine whether either roots of such order exist for equation (50).

In (50) we have:

$\Omega = 10000\, rad/s$;

$z_{0\varphi} \sim 0.0001\, m; \delta \sim 0.025\, m$ (accordingly with (40));

$g = 10\, m/s^2$;

besides, we put $r \sim 0.5\, m$.

In Figures 10-12 the roots of (50) are shown versus an immersion depth of the disk H.

So, we see that at *H = 1 m*, there are two appropriate roots near *z = 0.025 m*; at *H = 20 m* there is one appropriate root equal to exactly *0.025 m*; and at *H = 50 m*, there are already no

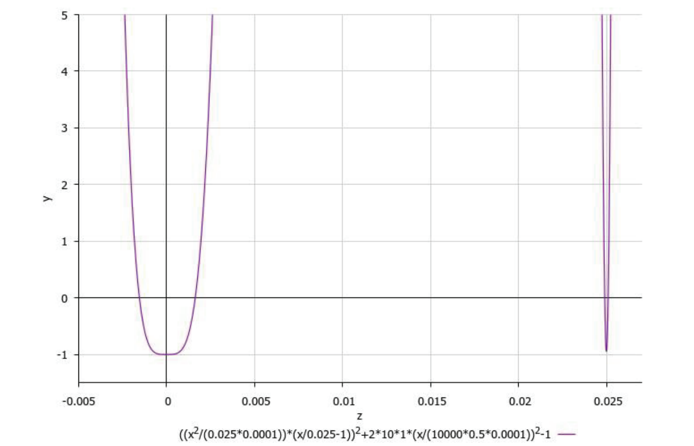

Figure 10: Roots of equation (50) at *H = 1 m*.

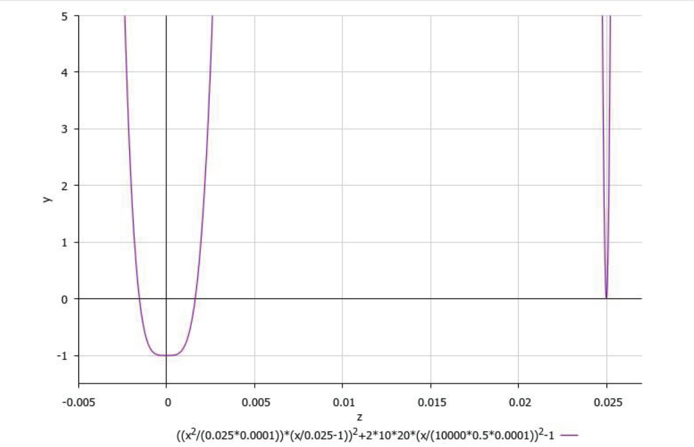

Figure 11: Roots of equation (50) at *H = 20 m*.

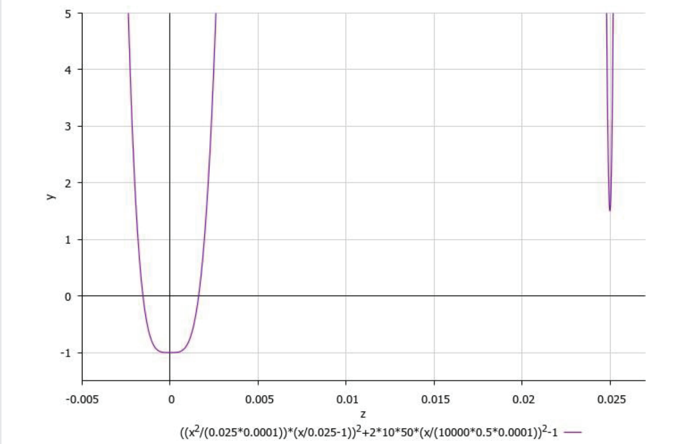

Figure 12: Roots of equation (50) at *H = 50 m*.

appropriate roots. Therefore, adjusting the magnitude of H we can always find solutions of (50) in the region of the turbulent layer.

Discussion

Implications of the findings

In this article, the solution implies obtaining the analytical expressions (closed-form formulae, not power series) for three components of velocity and pressure in cylindrical polar coordinates. Numerical methods are not applicable. The formulae for the velocity components are derived from the relations of tunnel mathematics since the latter is satisfied by all vector fields (including those obeying the Navier-Stokes equations). In this case, unnecessary solutions are inevitably obtained and must be cut off by means of a physical analysis of the task. The Navier-Stokes equation is then used for verification of obtained solutions and calculation of the pressure. Obtained formulae for pressure allow us to visualize the presence of the boundary layer and estimate its thickness for laminar and turbulent flows.

Limitations of the results

This method is based on algebraic tunnel mathematics [9] and has its limitations. For example, the proposed form of the real part of the function w is inapplicable in the vicinity of the point $r=0$ since its graph goes to infinity (Figure 3). Besides, although the graph for the azimuthal velocity v_φ in (32) is similar to the graph of function G in Figure 1 and can be used to model a laminar boundary layer, for a turbulent boundary layer the graph of azimuthal velocity has a much more complicated form [11,12].

Validity of the proposed solution and its applications

Since this method allows us to fairly accurately estimate the order of thickness of the laminar and turbulent boundary layers, we can talk about its certain applicability for engineering applications. Of course, tensor tunnel mathematics, unlike algebraic one, has a wider application and allows us to more accurately model Karman`s solution for a laminar boundary

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layer over a rotating disk [13]. Also currently used is the modeling of laminar flows over a rotating disk using the power series [14].

Conclusion

The solution of the Navier–Stokes equations using the tunnel mathematics apparatus is simple and elegant and also requires good mathematical training and a deep physical analysis of the problem. This method does not require specific software and can be used for the primary analysis of hydrodynamic problems. Obtained formulae for pressure allow us to visualize the presence of the boundary layer and estimate the order of its thickness for laminar and turbulent flows. The results obtained using this method for a fluid entrained by a disk of finite radius correspond to the results of a similar problem in classic handbooks on hydrodynamics where numerical methods were used.

Data availability

The data that supports the findings of this study are available within the article.

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