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Research Article

# Graceful Labeling of Posets

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## Abstract

The concept of graph labeling was introduced in the mid-1960s by Rosa. In this paper, we introduce a notion of graceful labeling of a finite poset. We obtain graceful labeling of some postes such as a chain, a fence, and a crown. In 2002 Thakare, Pawar, and Waphare introduced the `adjunct' operation of two lattices with respect to an adjunct pair of elements. We obtain the graceful labeling of an adjunct sum of two chains with respect to an adjunct pair (0, 1).

**AMS Subject Classification 2020:** 06A05, 06A06, 05C78

## Introduction

A graph labeling assigns integers to the vertices or edges (or both), subject to certain conditions. Interest in graph labeling began in the mid-1960s with the conjecture by Kotzing - Ringel [1] and a paper by Rosa [2]. There are different types of graph labeling such as prime labeling, magic labeling, antimagic labeling, graceful labeling [3], etc. Labeled graphs have wide applications in different fields such as circuit design, traffic control systems, communication network addressing, Automated Teller Machine (ATM) controlling devices, Local Area Network (LAN) network, radio astronomy, and Multiprotocol Label Switching (MPLS) protocols see [4-7]. In this paper, we define graceful labeling of finite posets. We obtain in particular graceful labeling of some posets like a chain, a fence, and a crown. Thakare, Pawar, and Waphare [8] introduced the `adjunct' operation of two lattices with respect to a pair of elements. In this connection, We obtain the graceful labeling of an adjunct sum of two chains concerning an adjunct pair (0, 1).

A non-empty set P, together with a binary relation  that is reflexive, antisymmetric, and transitive is called a *partially ordered set or a Poset*. A Hasse diagram is a type of mathematical diagram used to represent a finite partially ordered set. Specifically, for a poset (P, ≤) each element of P represents a vertex in the plane, and whenever y covers x, it indicates that $x \leq y$ and there is no z such that $x < z < y$, which is represented by $x \prec y$. These curves (or lines) may cross each other but must not touch any vertex other than endpoints; we call such curves (or lines) as edges. Two elements $a, b \in P$ are said to be *comparable* if either $a \leq b$ *or* $b \leq a$; otherwise they are said to be *incomparable*. A poset in which every pair of elements is comparable is called a *chain*. A chain on n elements is denoted by $C_n$. In particular, see Figure 1 for $C_3$.

**Definition 1** [9] *A partially ordered set* $F_n = \{x_1, x_2, \ldots, x_n\}$ *is called a fence (of order n ≥ 3), if either* $x_1 < x_2$, $x_2 > x_3$, , $x_{2m-1} < x_{2m}$



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, $x_{2m} > x_{2m+1}$, , $x_{n-1} < x_n$, *if n is even (* $x_{n-1} > x_n$ *if n is odd) or* $x_1 > x_2$, $x_2 < x_3$, , $x_{2m-1} > x_{2m}$, $x_{2m} < x_{2m+1}$, , $x_{n-1} > x_n$, *if n is even (* $x_{n-1} < x_n$ *if n is odd) are the only comparability relations. A fence $F_n$ is called a lower fence if* $x_1 < x_2$, *and upper fence if* $x_1 > x_2$. *In particular, see* Figure *1 for $F_3$ and $F_4$.*

**Definition 2** [10] *A crown is a poset* $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ *of order* $n \geq 2$, *whose elements satisfy precisely the comparabilities* $x_1 < y_1$, $y_1 > x_2$, $x_2 < y_2$, $y_2 > x_3$, $x_3 < y_3$, $y_3 > x_4$, , $x_{n-1} < y_{n-1}$, $y_{n-1} > x_n$, $x_n < y_n$, $y_n > x_1$. *The crown of order n is denoted by* $\mathbb{C}_n$. *In particular, see* Figure *1 for* $\mathbb{C}_4$.

For other definitions, notation, and terminology, see [11-13]. In the following section, we introduce the notion of graceful labeling of a poset.

## 2. Graceful labeling of posets

On the line of graceful labeling of graphs, we define graceful labeling of a finite poset as follows.

**Definition 3** *Let P be a poset on n elements with m coverings,* $x_1, x_2, \ldots, x_n$. *Let V =* $\{x_1, x_2, \ldots, x_n\}$ *and E =* $\{0, 1, 2, 3, \ldots, m\}$. *If* $\phi : V \to E$ *is a one-to-one function, then when each covering, say* $x_i \prec x_j$, *is given the label* $|\phi(x_i) - \phi(x_j)|$, *the resulting cover labels are unique numbers from the set E. This is known as the graceful labeling of P. A poset is called graceful if it has a graceful labeling. For example, $C_3$, $F_4$ and* $\mathbb{C}_4$ *are graceful* (Figure *2*).

**Theorem 2.1** *A chain $C_n$ is graceful for* $n \geq 2$.

*Proof.* Let $C_n : x_1 \prec x_2 \prec \cdots \prec x_n$ be a chain. Note that $C_n$ contains n - 1 edges. Let V = $\{x_1, x_2, \ldots, x_n\}$ be the set of elements of $C_n$ and E = {0, 1, 2, ..., $n-1$}. Define a map $\phi : V \to E$ as follows.

$$\phi(x_i) = \begin{cases} \frac{i-1}{2}, & \text{if i is odd} \\ n - \frac{i}{2}, & \text{if i is even.} \end{cases}$$

We claim that the map φ is the required graceful labeling of $C_n$. Firstly we prove that φ is one - one. One of the following four cases occurs. $\phi(x_i) = \phi(x_j)$ and both i and j are odd. But then $\frac{i-1}{2} = \frac{j-1}{2}$ which implies that i = j and hence $x_i = x_j$.

$\phi(x_i) = \phi(x_j)$ and both i and j are even. But then $n - \frac{i}{2} = n - \frac{j}{2}$ implies that i = j and hence $x_i = x_j$.

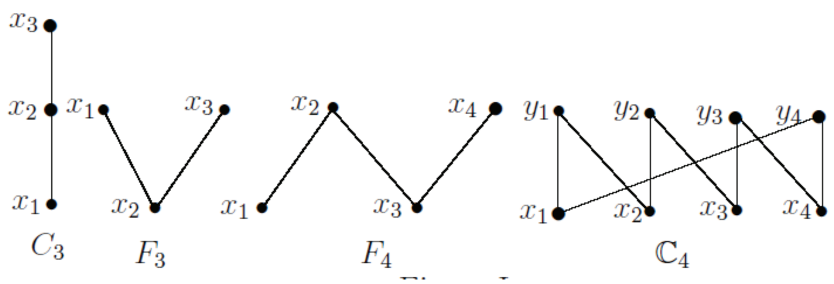


**Figure 1:** For C3. A chain on n elements is denoted by Cn. In particular.

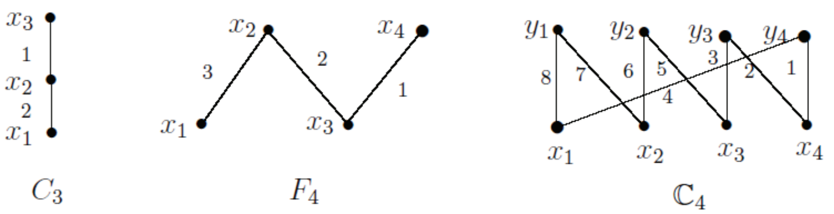


**Figure 2:** A poset is called graceful if it has a graceful labeling.



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i is odd and j is even. Then i $\neq$ j and $x_i \neq x_j \Rightarrow \phi(x_i) \neq \phi(x_j)$. For if, suppose $\phi(x_i) = \phi(x_j) \Rightarrow \frac{i-1}{2} = n - \frac{j}{2} \Rightarrow i-1 = 2n - j \Rightarrow i + j = 2n+1$. This is not possible, since $1 \le i \le n$ and $1 \le j \le n$.

i is even and j is odd. In this case, we get the proof on similar lines of Case (3). Thus, φ is one - one.

Secondly, we prove that the edge labels of $C_n$ are all distinct. Now the edge label between the elements $x_i$ and $x_{i+1}$ is given by $|\phi(x_{i+1}) - \phi(x_i)|$.

Suppose for $i \neq j$, $|\phi(x_{i+1}) - \phi(x_i)| = |\phi(x_{j+1}) - \phi(x_j)|$. One of the following three cases occurs.

1. Both i and j are odd. Then we have $|n - (\frac{i+1}{2}) - (\frac{i-1}{2})| = |n - (\frac{j+1}{2}) - (\frac{j-1}{2})|$. This implies that $|n-i| = |n-j|$ and hence $i = j$, which is a contradiction.
2. Both i and j are even. Then we have $|\frac{(i+1)-1}{2} - (n - \frac{i}{2})| = |\frac{(j+1)-1}{2} - (n - \frac{j}{2})|$. This implies that $|n-i| = |n-j|$ and hence $i = j$ which is a contradiction.
3. Without loss of generality, if i is even and j is odd, then we have $|\frac{(i+1)-1}{2} - (n - \frac{i}{2})| = |n - \frac{(j+1)}{2} - (\frac{j-1}{2})|$. This implies that $|i-n| = |n-j|$ and hence $i = j$, which is a contradiction. Hence the edge labels of $C_n$ are distinct.

Therefore φ is required graceful labeling $C_n$.

**Remark 1** *Let* $C_n : x_0 \prec x_1 \prec \cdots \prec x_{n-1}$ *be a chain where* $n \ge 2$. *Define a function* $\psi : V(C_n) \to \{0,1,2,\ldots,n-1\}$ *as follows.*

1. If n is odd

$$\psi(x_i) = \begin{cases} (n-1) - \lfloor \frac{n-i}{2} \rfloor, & \text{if i is even} \\ \frac{n-i}{2} - 1, & \text{if i is odd.} \end{cases}$$

2. If n is even

$$\psi(x_i) = \begin{cases} (n-1) - \lfloor \frac{n-i}{2} \rfloor, & \text{if i is odd} \\ \frac{n-i}{2} - 1, & \text{if i is even.} \end{cases}$$

Then Ψ is also a graceful labeling of $C_n$.

By the arguments similar to one given in the proof of Theorem 2.1, we obtain the proof of the following result, since, $F_n$ the edge labels are the same as that of the chain $C_n$.

**Corollary 2.2** *A fence $F_n$ is graceful for n $\ge$ 3.*

Note that, the graceful labeling of a chain on n elements and a fence on n elements are the same. Therefore, we have the following.

**Theorem 2.3** *A crown $\mathbb{C}_n$ is graceful if n is even.*

*Proof.* Suppose the set of elements of crown $\mathbb{C}_n$ is V $= \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ with 2n coverings $x_1 \prec y_1, x_2 \prec y_1, x_2 \prec y_2, x_3 \prec y_2, \cdots, x_{n-1} \prec y_{n-1}, x_n \prec y_{n-1}, x_n \prec y_n, x_1 \prec y_n$. Let E $= \{0,1,2,\ldots,2n\}$. Define a map $\phi : V \to E$ as follows.

$\phi(x_i) = i-1$, if $1 \le i \le n$, and $\phi(y_i) = \begin{cases} 2n-(i-1), & \text{if } 1 \le i \le \frac{n}{2}. \\ 2n-i, & \text{if } \frac{n}{2}+1 \le i \le n. \end{cases}$

We claim that the map is a graceful labeling of $\mathbb{C}_n$. Firstly we prove that $\varphi$ is one-one. One of the following five cases occurs.

1. $\phi(x_i) = \phi(x_j)$ and $1 \le i, j \le n$. Then $i - 1 = j - 1$ implies that $i$ $j$ and hence $x_i = x_j$.

2. Suppose that $\phi(y_i) = \phi(y_j)$ and $1 \le i, j \le \frac{n}{2}$. Then $2n-(i-1) = 2n-(j-1)$ implies that $i - j$ and hence $y_i = y_j$.



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3. that $\phi(y_i)=\phi(y_j)$ and $\frac{n}{2}+1\le i,j\le n$. Then $2n-i=2n-j$ implies that $i-j$ and hence $y_i=y_j$.

4. $1\le i\le\frac{n}{2}$ and $x_i\ne y_i$. Then $\phi(x_i)\ne\phi(y_i)$. For if, suppose $\phi(x_i)=\phi(y_i)$ implies that $i-1=2n-(i-1)\Rightarrow 2i=2n+2\Rightarrow i=n+1$ which is not possible.

5. $\frac{n}{2}+1\le i\le n$ and $x_i\ne y_i$. Then $\phi(x_i)\ne\phi(y_i)$. Since, $\phi(x_i)=\phi(y_i)$ then $i-1=2n-i\Rightarrow 2i=2n+1$, which is not possible.

Thus $\varphi$ is one - one.

Secondly, we prove the edge labels of $\mathbb{C}_n$ are all distinct. Consider the edge labels of $\mathbb{C}_n$ for $1\le i\le n$ as $|\phi(x_i)-\phi(y_i)|$, for $2\le i\le n$ as $|\phi(x_i)-\phi(y_{i-1})|$, and $|\phi(y_n)-\phi(x_1)|$. One of the following five cases occurs.

1. $1\le i,k\le\frac{n}{2}$ and $i\ne k$.

Suppose $|\phi(x_i)-\phi(y_i)|=|\phi(x_k)-\phi(y_k)|\Rightarrow|i-1-(2n-(i-1))|=|k-1-(2n-(k-1))|\Rightarrow|-2n+2i-2|=|-2n+2k-2|\Rightarrow i=k$ which is a contradiction. Now let $\frac{n}{2}+1\le i,k\le n$ and $i\ne k$. Suppose $|\phi(x_i)-\phi(y_i)|=|\phi(x_k)-\phi(y_k)|\Rightarrow|i-1-(2n-i)|=|k-1-(2n-k)|\Rightarrow|-2n+2i-1|=|-2n+2k-1|\Rightarrow i=k$ which is a contradiction.

2. $2\le i,k\le\frac{n}{2}$ and $i\ne k$.

Suppose

$|\phi(x_i)-\phi(y_{i-1})|=|\phi(x_k)-\phi(y_{k-1})|\Rightarrow|i-1-(2n-((i-1)-1))|=|k-1-(2n-(k-1))|\Rightarrow|-2n+2i-3|=|-2n+2k-3|\Rightarrow i=k$

which is a contradiction. Now let $\frac{n}{2}+1\le i,k\le n$ and $i\ne k$. Suppose $|\phi(x_i)-\phi(y_{i-1})|=|\phi(x_k)-\phi(y_{k-1})|\Rightarrow|i-1-(2n-(i-1))|=|k-1-(2n-(k-1)|\Rightarrow|-2n+2i-2|=|-2n+2k-2|\Rightarrow i=k$ which is a contradiction.

$1\le i\le\frac{n}{2}$ and suppose $|\phi(x_i)-\phi(y_i)|=|\phi(x_1)-\phi(y_n)|\Rightarrow|i-1-(2n-(i-1))|=|1-1-(2n-(n-1))|\Rightarrow|2i-2n-2|=|-n+1|\Rightarrow 2i=n+3$ which is not possible. Let $\frac{n}{2}+1\le i\le n$ and suppose $|\phi(x_i)-\phi(y_i)|=|\phi(x_1)-\phi(y_n)|\Rightarrow|i-1-(2n-(i))|=|1-1-(2n-(n))|\Rightarrow|2i-2n-1|=|-n|\Rightarrow 2i-1=3n$ which is not possible.

$2\le i\le\frac{n}{2}$ and suppose

$|\phi(x_i)-\phi(y_{i-1})|=|\phi(x_1)-\phi(y_n)|\Rightarrow|i-1-(2n-((i-1)-1))|=|1-1-(2n-(n))|\Rightarrow|i-1-2n+i-2|=|-n|\Rightarrow 2i-2n-3=n\Rightarrow 2i-3=3n$ whichisnot possible. Let $\frac{n}{2}+1\le i\le n$ and suppose $|\phi(x_i)-\phi(y_{i-1})|=|\phi(x_1)-\phi(y_n)|\Rightarrow|i-1-(2n-(i-1))|=|1-1-(2n-n)|\Rightarrow|2i-2n-2|=|-n|\Rightarrow 2i-2=3n$ which is not possible.

$1\le i\le\frac{n}{2}$ and $2\le k\le\frac{n}{2}$, suppose

$|\phi(x_i)-\phi(y_i)|=|\phi(x_k)-\phi(y_{k-1})|\Rightarrow|i-1-(2n-(i-1))|=|k-1-(2n-((k-1)-1)|\Rightarrow|2i-2n-2|=|2k-2n-3|\Rightarrow 2i-2=2k-3\Rightarrow 2i-2k=-1\Rightarrow i-k=\frac{-1}{2}$ which is not possible.

Here, the map $\varphi$ gives the required graceful labeling of the $\mathbb{C}_n$.

**Theorem 2.4** *A crown $\mathbb{C}_n$ has no graceful labeling if n is odd.*

*Proof.* Let $\mathbb{C}_n$ be crown with the set of elements $V=\{x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n\}$ with 2n Coverings $x_1\prec y_1,x_2\prec y_1,x_2\prec y_2,x_3\prec y_2,\cdots,x_{n-1}\prec y_{n-1},x_n\prec y_{n-1},x_n\prec y_n,x_1\prec y_n$. Let $E=\{0,1,2,\ldots,2n\}$. Suppose $\mathbb{C}_n$ has p - labeling as $\phi:V\to E$. Taking the sum of edge labels of the crown $\mathbb{C}_n$. We have $0\le(|\phi(x_1)-\phi(y_1)|+|\phi(y_1)-\phi(x_2)|+|\phi(x_2)-\phi(y_2)|+\ldots+|\phi(x_{n-1})-\phi(y_{n-1})|+|\phi(x_n)-\phi(y_{n-1})|+|\phi(x_n)-\phi(y_n)|+|\phi(x_1)-\phi(y_n)|)$

$\le(|\phi(x_1)|+|\phi(y_1)|+|\phi(y_1)|+|\phi(x_2)|+|\phi(x_2)|+|\phi(y_2)|+\ldots+|\phi(x_{n-1})|+|\phi(y_{n-1})|+|\phi(x_n)|+|\phi(y_{n-1})|+|\phi(x_n)|+|\phi(y_n)|+|\phi(x_1)|+|\phi(y_n)|)$ $=$ $2(|\phi(x_1)|+|\phi(x_2)|+\ldots+|\phi(x_n)|+|\phi(y_1)|+|\phi(y_2)|+\ldots+|\phi(y_n)|)$.



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Therefore,

$$(|\phi(x_1)-\phi(y_1)|+|\phi(y_1)-\phi(x_2)|+|\phi(x_2)-\phi(y_2)|+\ldots+|\phi(x_{n-1})-\phi(y_{n-1})|+|\phi(x_n)-\phi(y_{n-1})|+|\phi(x_n)-\phi(y_n)|+|\phi(x_1)-\phi(y_n)|)\equiv 0(\bmod 2)$$

In the crown $\mathbb{C}_n$, edge labels are from 1 to 2n. So, the sum of edge labels of $\mathbb{C}_n$ is

$\sum_{k=1}^{2k} k=\frac{2n(2n+1)}{2}=n(2n+1)$ which is odd. As $n$ is odd, $2n$+1 is odd, and therefore $n(2n+1)$ must be odd. Therefore we get a contradiction. Hence we conclude that $\mathbb{C}_n$ has no graceful labeling, if n odd.

## 3. Adjunct sum of lattices

In 2002, Thakare, Pawar, and Waphare [8] introduced the concept of an adjunct sum of lattices.

**Definition 4** *[8] Suppose $L_1$ and $L_2$ are two disjoint lattices and (a,b) is a pair of elements in $L_1$ such that a<b and* $a \not\prec b$. *Define the partial order* $\le$ *on* $L=L_1\cup L_2$ *with respect to the pair (a,b) as follows:* $x\le y$ *in L if* $x,y\in L_1$ *and* $x\le y$ *in $L_{1'}$ or* $x,y\in L_2$ *and* $x\le y$ *in* $L_2$, *or* $x\in L_1$, $y\in L_2$ *and* $x\le a$ *in $L_{1'}$ or* $x\in L_2$, $y\in L_1$ *and* $b\le y$ *in $L_{1'}$.*

It is easy to see that $L$ is a lattice containing $L_1$ and $L_2$ as sublattices. The procedure for obtaining $L$ in this way is called *adjunct operation (or adjunct sum)* of $L_1$ with $L_2$. We call the pair *(a,b)* as an *adjunct pair* and $L$ as an *adjunct* of $L_1$ with $L_2$ concerning the adjunct pair *(a,b)* and write $L=L_1]_a^b L_2$. A diagram of $L$ is obtained by placing a diagram of $L_1$ and a diagram of $L_2$ side by side in such a way that the largest element 1 of $L_2$ is at lower position than $b$ and the least element $0$ of $L_2$ is at the higher position than $a$ and then by adding the coverings *<1, b>* and *<a, 0>*, as shown in Figure III. This gives $|E(L)|=|E(L_1)|+|E(L_2)|+2$.

The adjunct sum is often utilized to construct and analyze complex lattices from simpler, well-defined components while retaining the essential properties of a lattice. To obtain graceful labeling for lattices formed by the adjunct sum of two chains, we construct the following sets. Let A = $\{1,3,5,\ldots,m-1\}, m\ge 3$, $B'=\{2,4,6,\ldots,\frac{m+n+1}{2}\}, B''=\{\frac{m+n+1}{2}+2,\frac{m+n+1}{2}+4,\ldots,m\}$ and $B=B'\cup B''$. Also, let $D=\{1,3,5,\ldots,n\}, n\ge 1, F'=\{2,4,6,\ldots,\frac{m+n+1}{2}-2\}, F''=\{\frac{m+n+1}{2},\frac{m+n+1}{2}+2,\ldots,n-1\}$, and $F=F'\cup F''$.

**Theorem 3.1** *Let C and C′ be the chains with* $|C|=m\ge$ *3,* $|C'|=n\ge 1$ *and* $L=C]_0^1 C'$. *Then L has graceful labeling if m ≡ 2(mod 4) and n  1(mod 4).*

*Proof.* Let C and $C'$ be the chains with m and n elements, respectively. Suppose *m ≡ 2(mod 4) and n  1(mod 4)* and. Suppose $C=a_1\prec a_2\prec a_3\ldots\prec a_m$, $C'=b_1\prec b_2\prec b_3\ldots\prec b_n$ and $L=C]_0^1 C'$. Clearly, L has *m+n* elements and *m+n* coverings (edges). Suppose V = $\{a_1,a_2,a_3,\ldots,a_m,b_1,b_2,b_3,\ldots,b_n\}$ and E = {0, 1, 2, … , m+n}. Consider a map $\phi: V\to E$ defined as follows :

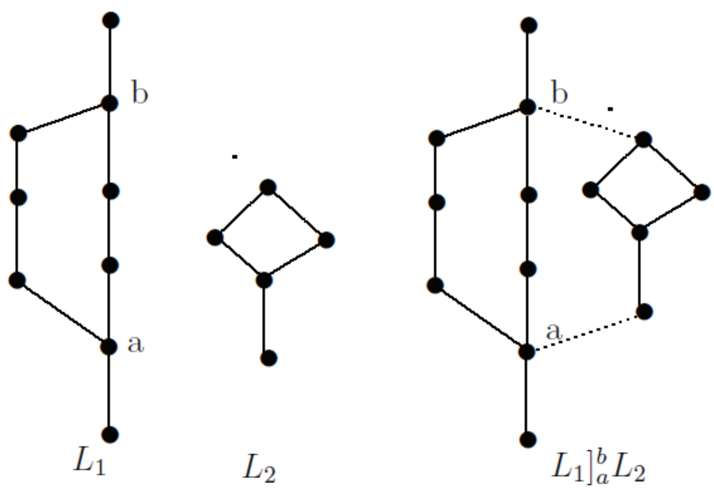


**Figure 3:** A diagram of L is obtained by placing a diagram of L1 and a diagram of L2.



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$$\phi(a_i)=\begin{cases}\frac{i-1}{2} & \text{if} \quad i\in A\\ m+n-\left\lfloor\frac{i-1}{2}\right\rfloor & \text{if} \quad i\in B'\\ m+n-\frac{i}{2} & \text{if} \quad i\in B''\end{cases}$$

$$\phi(b_j)=\begin{cases}\frac{m+n-j}{2} & \text{if} \quad j\in D\\ \frac{m+n+j-1}{2} & \text{if} \quad j\in F'\\ \frac{m+n+j+1}{2} & \text{if} \quad j\in F''\end{cases}$$

We claim that the map ϕ is the required graceful labeling for lattice L. Firstly we prove that $\varphi$ is one - one. For this purpose, we consider the following sets. Let $S_1=\{a_i \mid i\in A\}$, $S_2'=\{a_i \mid i\in B'\}$, $S_2''=\{a_i \mid i\in B''\}$ and $S_2=S_2'\cup S_2''$. Also, let $T_1=\{b_j \mid j\in D\}$, $T_2'=\{b_j \mid j\in F\}$, $T_2''=\{b_j \mid j\in F''\}$ and $T_2=T_1'\cup T_2''$. Let $a, b \in V$. Now to show $\varphi$ is one-one. For this proof, one of the following five cases occurs depending on $a, b \in S_k$ (k = 1, 2) and $a, b \in T_k$ (k =1, 2):

1) Suppose $a, b \in S_k$ (k = 1, 2).

a) Suppose $a, b \in S_1$. Therefore $a = a_i$ and $b = a_j$ for $i, j \in A$. Consider $\phi(a)=\phi(b)$ i.e. $\phi(a_i)=\phi(a_j)$. $\Rightarrow \frac{(i-1)}{2}=\frac{(j-1)}{2}$ for $i, j \in A$. $\Rightarrow i=j \ \Rightarrow a_i=a_j$ i.e $a = b$.

b) Let $a, b \in S_2$, here we have three parts.

I. Suppose $a, b \in S'_2$. Therefore $a = a_i$ and $b = a_j$ for $i,j\in B'$. Consider $\phi(a)=\phi(b)$ i.e. $\phi(a_i)=\phi(a_j)$. $\Rightarrow m+n-\lfloor\frac{(i-1)}{2}\rfloor=m+n-\lfloor\frac{(j-1)}{2}\rfloor$ for $i,j\in B'$. $\Rightarrow\lfloor\frac{(i-1)}{2}\rfloor=\lfloor\frac{(j-1)}{2}\rfloor \ \Rightarrow i=j\Rightarrow a_i=a_j$ i.e. $a = b$.

II. Suppose $a, b \in S''_2$. Therefore $a = a_i$ and $b = a_j$ for $i,j\in B''$. Consider $\phi(a)=\phi(b)$ i.e. $\phi(a_i)=\phi(a_j)$. $m+n-\frac{i}{2}=m+n-\frac{j}{2}$ $\Rightarrow i=j\Rightarrow a_i=a_j$ i.e. $a = b$.

III. Without loss of generality suppose that $a \in S'_2$ and $b \in S''_2$. Therefore $a = a_i$ and $b = a_j$ for $i\in B'$ and $j \in B''$. Claim: If $a\neq b$ then $\phi(a)\neq\phi(b)$. Suppose $a\neq b$. For if suppose $\phi(a)=\phi(b)$ i.e. $\phi(a_i)=\phi(a_j)$. Therefore $m+n-\lfloor\frac{(i-1)}{2}\rfloor=m+n-\frac{j}{2}$. $\Rightarrow\lfloor\frac{(i-1)}{2}\rfloor=\frac{j}{2}$. Which is not possible, since $\lfloor\frac{(i-1)}{2}\rfloor<\frac{j}{2}$ for $i\in B'$ and $j\in B''$. Thus, $\phi(a_i)\neq\phi(a_j)$ i.e. $\phi(a)\neq\phi(b)$.

c) Without loss of generality suppose that $a \in S_1$ and $b \in S'_2$, then we have the following two cases.

I. Let $a \in S_1$ and $b \in S'_2$. Therefore $a = a_i$ and $b = a_j$ for $i$ $A$ and $j\in B'$. Claim: $a\neq b$ then $\phi(a)\neq\phi(b)$. Suppose $a\neq b$. For if suppose $\phi(a)=\phi(b)$ i.e. $\phi(a_i)=\phi(a_j)$. Therefore $\frac{i-1}{2}=m+n-\lfloor\frac{j-1}{2}\rfloor$. i.e. $m+n=\frac{i-1}{2}+\lfloor\frac{j-1}{2}\rfloor$. Which is not possible, since $m+n>\frac{i-1}{2}+\lfloor\frac{j-1}{2}\rfloor$, as $i\leq m-1$ and $j\leq\frac{m+n+1}{2}$. Thus, $\phi(a_i)\neq\phi(a_j)$ i.e. $\phi(a)\neq\phi(b)$.

II. Let $a \in S_1$ and $b \in S''_2$. Therefore $a = a_i$ and $b = a_j$ for $i$ $A$ and $j\in B''$. Claim: If $a\neq b$ then $\phi(a)\neq\phi(b)$. Suppose $a\neq b$ For, if suppose $\phi(a)=\phi(b)$ i.e. $\phi(a_i)=\phi(a_j)$. Therefore $\frac{i-1}{2}=m+n-\frac{j}{2}$. i.e. $i+j-1=2(m+n)$. Which is not possible since, $2(m+n)>i+j-1$ as $i\leq m-1, j\leq n$. Thus, $\phi(a_i)\neq\phi(a_j)$ i.e. $\phi(a)\neq\phi(b)$.

2. Suppose $a,b \in T_k$ (k = 1, 2.)

a) Suppose $a,b \in T_1$. Therefore $a = b_i$ and $b = b_j$ for $i, j \in D$. Suppose $\phi(a)=\phi(b)$ i.e. $\phi(b_i)=\phi(b_j)$. Therefore $\frac{m+n-i}{2}=\frac{m+n-j}{2}$. $\Rightarrow i=j\Rightarrow b_i=b_j$ i.e. $a = b$.

b) Suppose $a,b \in T_2$, here we have three parts.



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I. Suppose $a,b \in T'_2$. Therefore $a = b_i$ and $b = b_j$ for $i, j \in F'$. Consider $\phi(a)=\phi(b)$ i.e. $\phi(b_i)=\phi(b_j)$. $\Rightarrow \frac{m+n+i-1}{2}=\frac{m+n+j-1}{2}$.

$\Rightarrow i=j \Rightarrow b_i=b_j$ i.e. $a = b$.

II. Suppose $a,b \in T''_2$. Therefore $a = b_i$ and $b = b_j$ for $i, j \in F''$. Suppose $\phi(a)=\phi(b)$ i.e. $\phi(b_i)=\phi(b_j)$. $\Rightarrow \frac{m+n+i+1}{2}=\frac{m+n+j+1}{2}$

$\Rightarrow i=j \Rightarrow b_i=b_j$ i.e. $a = b$.

III. Without loss of generality $a \in T'_2$ and $b \in T''_2$. Therefore $a = b_i$ and $b = b_j$ for some $i \in F'$ and $j$ $F''$. Claim: $a \neq b$ then $\phi(a)\neq\phi(b)$. Suppose $a \neq b$ For, if suppose $\phi(a)=\phi(b)$ i.e. $\phi(b_i)=\phi(b_j)$. Therefore $\frac{m+n+i-1}{2}=\frac{m+n+j+1}{2}$. i.e. $i-1=j+1 \Rightarrow i=j+2$. Which is not possible since $j > i$ as $i \leq \frac{m+n+1}{2}-2$, $j \leq n-1$. $\Rightarrow \phi(b_i)\neq\phi(b_j)$ i.e. $\phi(a)\neq\phi(b)$.

c) Without loss of generality, let $a \in T_1$ and $b \in T_2$, then we have the following two parts.

I. Suppose $a \in T_1$ and $b \in T'_2$. Therefore $a = b_i$ and $b = b_j$ for some $i \in D$ and $j \in F'$. Claim: If $a \neq b$ then $\phi(a)\neq\phi(b)$. Suppose $a \neq b$ For, if suppose $\phi(a)=\phi(b)$ i.e. $\phi(b_i)=\phi(b_j)$. $\Rightarrow \frac{m+n-i}{2}=\frac{m+n+j-1}{2}$ $\Rightarrow -i=j-1$ which is not possible since $i$ $D$ and $j \in F'$. $\Rightarrow \phi(b_i)\neq\phi(b_j)$ i.e. $\phi(a)\neq\phi(b)$.

II. Suppose $a \in T_1$ and $b \in T''_2$. Therefore $a = b_i$ and $b = b_j$ for some $i \in D$ and $j \in F'$. Claim: If $a \neq b$ then $\phi(a)\neq\phi(b)$. Suppose $a \neq b$. For, if suppose $\phi(a)=\phi(b)$ i.e. $\phi(b_i)=\phi(b_j)$. Therefore $\frac{m+n-i}{2}=\frac{m+n+j+1}{2}$ i.e. $-i=j+1$, which is not possible since $i$ $D$ and $j \in F''$. $\Rightarrow \phi(b_i)\neq\phi(b_j)$ i.e $\phi(a)\neq\phi(b)$.

3. Let $a \in S_k$ (k = 1,2) and $b \in T_k$ (k = 1, 2).

a) Suppose $a \in S_1$ and $b \in T_1$. Therefore $a = a_i$ and $b = b_j$ for some $i \in A$ and $j \in D$. Claim: If $a \neq b$ then $\phi(a)\neq\phi(b)$. Suppose $a \neq b$ For, if suppose $\phi(a)=\phi(b)$ i.e. $\phi(a_i)=\phi(b_j)$. Therefore $\frac{i-1}{2}=\frac{m+n-j}{2}$. $\Rightarrow m+n=i+j-1$ which is not possible since $m+n > i+j-1$, since $i \leq m-1$, and $j \leq n$. Thus, $\phi(a_i)\neq\phi(b_j)$ i.e. $\phi(a)\neq\phi(b)$.

b) Suppose $a \in S_1$ and $b \in T_2$, then we have the following two parts.

I. Suppose $a \in S_1$ and $b \in T'_2$ and. Therefore $a = a_i$ and $b = b_j$ for $i \in A$ and $j \in F'$. Claim: If $a \neq b$ then $\phi(a)\neq\phi(b)$. Suppose $a \neq b$. For, if suppose $\phi(a)=\phi(b)$ i.e. $\phi(a_i)=\phi(b_j)$. $\Rightarrow \frac{i-1}{2}=\frac{m+n+j-1}{2}$. $\Rightarrow m+n=i-j$ which is not possible since $m+n > i-j$, since $i \leq m-1$ and $j \leq \frac{m+n1}{2}-2$. Thus, $\phi(a_i)\neq\phi(b_j)$ i.e. $\phi(a)\neq\phi(b)$.

II. Suppose $a \in S_1$ and $b \in T''_2$. Therefore $a = a_i$ and $b = b_j$ for $i \in A$ and $j \in F''$. Claim: If $a \neq b$ then $\phi(a)\neq\phi(b)$. Suppose $a \neq b$ For, if suppose $\phi(a)=\phi(b)$ i.e. $\phi(a_i)=\phi(b_j)$. $\Rightarrow \frac{i-1}{2}=\frac{m+n+j+1}{2}$ $\Rightarrow m+n=i-j$ which is not possible since $m+n > i-j-2$, as $i \leq m-1$ and $j \leq n-1$. $\Rightarrow \phi(a_i)\neq\phi(b_j)$ i.e. $\phi(a)\neq\phi(b)$.

c) Suppose $a \in S_2$ and $b \in T_2$ and then we have the following parts.

I. Suppose $a \in S'_2$ and $b \in T'_2$. Therefore $a = a_i$ and $b = b_j$ for some $i$ $B'$ and $j \in F'$. Claim: If $a \neq b$ then $\phi(a)\neq\phi(b)$. Suppose $a \neq b$. For, if suppose $\phi(a)=\phi(b)$ i.e. $\phi(a_i)=\phi(b_j)$. Therefore $m+n-\lfloor\frac{(i-1)}{2}\rfloor=\frac{m+n+j-1}{2}$. $\Rightarrow 2(m+n)-2(\lfloor\frac{(i-1)}{2}\rfloor)=m+n+j-1$. $\Rightarrow m+n=j+2(\lfloor\frac{(i-1)}{2}\rfloor)-1$ which is not possible since, $m+n > j-(\lfloor\frac{(i-1)}{2}\rfloor)-1$ since, $i \leq \frac{m+n+1}{2}$ and $j \leq \frac{m+n+1}{2}$. Thus $\phi(a_i)\neq\phi(b_j)$ i.e. $\phi(a)\neq\phi(b)$.

II. Suppose $a \in S'_2$ and $b \in T''_2$. Therefore $a = a_i$ and $b = b_j$ for $i \in B'$ and $j$ $F'$. Claim: If $a \neq b$ then $\phi(a)\neq\phi(b)$. Suppose



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$a \neq b$. For, if suppose $\phi(a)=\phi(b)$ i.e. $\phi(a_i)=\phi(b_j)$. $\Rightarrow m+n-\lfloor\frac{(i-1)}{2}\rfloor=\frac{m+n+j+1}{2}$. $\Rightarrow 2(m+n)-2(\lfloor\frac{(i-1)}{2}\rfloor)=m+n+j-1$. $m+n=j+2(\lfloor\frac{(i-1)}{2}\rfloor)+1$ which is not possible Since, $i \in B'$ and $j \in F'$. $\Rightarrow \phi(a_i) \neq \phi(b_j)$ i.e. $\phi(a) \neq \phi(b)$.

III. Suppose $a \in S''_2$ and $b \in T'_2$. Therefore $a = a_i$ and $b = b_j$ for $i \in B''$ and $j \in F'$. Claim: If $a \neq b$ then $\phi(a) \neq \phi(b)$. Suppose $a \neq b$. For, if suppose $\phi(a)=\phi(b)$ i.e. $\phi(a_i)=\phi(b_j)$. Therefore $m+n-\frac{i}{2} = \frac{m+n+j-1}{2}$. $\Rightarrow (m+n)=j+i-1$ which is not possible since, $i \in B''$ and $j \in F'$. Thus, $\phi(a_i) \neq \phi(b_j)$ i.e. $\phi(a) \neq \phi(b)$.

IV. Suppose $a \in S''_2$ and $b \in T''_2$. Therefore $a = a_i$ and $b = b_j$ for $i$ $B''$ and $j \in F''$. Claim: If $a \neq b$ then $\phi(a) \neq \phi(b)$. Suppose $a \neq b$. For, if suppose $\phi(a)=\phi(b)$ i.e. $\phi(a_i)=\phi(b_j.)$ Therefore $m+n-\frac{i}{2}=\frac{m+n+j+1}{2}$. $\Rightarrow (m+n)=j+i+1$ which is not possible since $i$ $B''$ and $j \in F''$. $\Rightarrow \phi(a_i) \neq \phi(b_j)$ i.e. $\phi(a) \neq \phi(b)$. Hence $\varphi$ is one - one function.

Secondly to show edge labels of L are distinct. We have edge labels of L are

$|\phi(a_i)-\phi(a_{i+1})|=m+n-(i-1)$ for $1 \le i \le m-2$.

$|\phi(a_i)-\phi(a_{i+1})|=m+n-(i)$ for $i=m-1$.

$|\phi(b_j)-\phi(b_{j+1})|=j$ for $1 \le j \le n-1$.

$|\phi(a_i)-\phi(b_j)|=\frac{(m+n-1)}{2}$ for $i=j=1$.

$|\phi(a_m)-\phi(b_j)|=n$ for i = m and j = n. From the above labeling pattern, it is observed that the edge labels of L are distinct. Thus, lattice L has graceful labeling.

Using proof of theorem 4.1, one can obtain the proof of the following theorems.

**Theorem 3.2** *Let C and* $C'$ *are chains with* $|C|=m \geq 3$ *and* $|C'|=n \geq 1$ *and* $L=C]_0^1C'$. *Then L has graceful labeling if m ≡ 3(mod 4) and n ≡ 1(mod 4).*

*Proof.* Let $C$ and $C'$ be the chains with m and n elements, respectively. Suppose *m ≡ 3(mod 4) and n ≡ 1(mod 4)*. Suppose $C=a_1 \prec a_2 \prec a_3 \ldots \prec a_m$, $C'=b_1 \prec b_2 \prec b_3 \ldots \prec b_n$ and $L=C]_0^1C'$. Clearly, L has *m+n* elements and *m+n* coverings (edges). Suppose V = $\{a_1,a_2,a_3,\ldots,a_m,b_1,b_2,b_3,\ldots,b_n\}$ and E = {0, 1, 2, ... , *m+n*}. Consider a map $\phi: V \to E$ defined as follows :

$$\phi(a_i)=\begin{cases} \frac{i-1}{2} & \text{if} \quad i=1,3,5,\ldots,m \\ m+n-\left\lfloor\frac{i-1}{2}\right\rfloor & \text{if} \quad i=2,4,6,\ldots,\frac{m+n}{2} \\ m+n-\frac{i}{2} & \text{if} \quad i=\frac{m+n}{2}+2,\frac{m+n}{2}+4\ldots,m-1 \end{cases}$$

$$\phi(b_j)=\begin{cases} \frac{m+n+j-1}{2} & \text{if} \quad j=1,3,5,\ldots,\frac{m+n}{2}-1 \\ \frac{m+n+j+1}{2} & \text{if} \quad j=\frac{m+n}{2}+1,\frac{m+n}{2}+3,\ldots,n \\ \frac{m+n-j}{2} & \text{if} \quad j=2,4,6,\ldots,n-1 \end{cases}$$

Clearly $\varphi$ gives the required graceful labeling for L.

**Theorem 3.3** *Let C and* $C'$ *are chains with* $|C|=m \geq 3$ *and* $|C'|=n \geq 1$ *and* $L=C]_0^1C'$. *Then L has graceful labeling if m ≡ 1(mod 4) and n ≡ 2(mod 4).*



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*Proof.* Let C and $C'$ be the chains with m and n elements, respectively. Suppose $m \equiv 1 (mod\ 4)$ and $n \equiv 3 (mod\ 4)$. Suppose $C = a_1 \prec a_2 \prec a_3 \ldots \prec a_m$, $C' = b_1 \prec b_2 \prec b_3 \ldots \prec b_n$ and $L = C]_0^1 C'$. Clearly, L has $m+n$ elements and $m+n$ coverings (edges). Suppose V = $\{a_1, a_2, a_3, \ldots, a_m, b_1, b_2, b_3, \ldots, b_n\}$ and E = {0, 1, 2, … , $m+n$}. Consider a map $\phi : V \to E$ defined as follows :

$$\phi(a_i) = \begin{cases} \frac{i-1}{2} & \text{if} \quad i = 1,3,5,\ldots,m \\ m+n-\left\lfloor \frac{i-1}{2} \right\rfloor & \text{if} \quad i = 2,4,6,\ldots,\frac{m+n+1}{2} \\ m+n-\frac{i}{2} & \text{if} \quad i = \frac{m+n+1}{2}+2, \frac{m+n+1}{2}+4, \ldots, m-1 \end{cases}$$

$$\phi(b_j) = \begin{cases} \frac{m+n-j}{2} & \text{if} \quad j = 1,3,5,\ldots,n-1 \\ \frac{m+n+j-1}{2} & \text{if} \quad j = 2,4,6,\ldots,\frac{m+n+1}{2}-2 \\ \frac{m+n+j+1}{2} & \text{if} \quad j = \frac{m+n+1}{2}, \frac{m+n+1}{2}+2, \ldots, n \end{cases}$$

Clearly $\varphi$ gives the required graceful labeling for L.

**Theorem 3.4** *Let C and $C'$ are chains with $|C| = m \geq 3$ and $|C'| = n \geq 1$ and $L = C]_0^1 C'$. Then L has graceful labeling if $m \equiv 2 (mod\ 4)$ and $n \equiv 2 (mod\ 4)$.*

*Proof.* Let C and $C'$ be the chains with m and n elements, respectively. Suppose $m \equiv 2 (mod\ 4)$ and $n \equiv 2 (mod\ 4)$. Suppose $C = a_1 \prec a_2 \prec a_3 \ldots \prec a_m$, $C' = b_1 \prec b_2 \prec b_3 \ldots \prec b_n$ and $L = C]_0^1 C'$. Here, L has $m+n$ elements and $m+n$ coverings (edges). Suppose V = $\{a_1, a_2, a_3, \ldots, a_m, b_1, b_2, b_3, \ldots, b_n\}$ and E = {0, 1, 2, … , $m+n$}. Consider a map $\phi : V \to E$ defined as follows :

$$\phi(a_i) = \begin{cases} \frac{i-1}{2} & \text{if} \quad i = 1,3,5,\ldots,m-1 \\ m+n-\left\lfloor \frac{i-1}{2} \right\rfloor & \text{if} \quad i = 2,4,6,\ldots,\frac{m+n}{2} \\ m+n-\frac{i}{2} & \text{if} \quad i = \frac{m+n}{2}+2, \frac{m+n}{2}+4, \ldots, m \end{cases}$$

$$\phi(b_j) = \begin{cases} \frac{m+n+j-1}{2} & \text{if} \quad j = 1,3,5,\ldots,m-1 \\ \frac{m+n+j+1}{2} & \text{if} \quad j = \frac{m+n}{2}+1, \frac{m+n}{2}+3, \ldots, n-1 \\ \frac{m+n-j}{2} & \text{if} \quad j = 2,4,6,\ldots,n \end{cases}$$

Clearly $\varphi$ gives the required graceful labeling for L.

**Theorem 3.5** *Let C and $C'$ are chains with $|C| = m \geq 3$ and $|C'| = n \geq 1$ and $L = C]_0^1 C'$. Then L has graceful labeling if $m \equiv 0 (mod\ 4)$ and $n \equiv 3 (mod\ 4)$.*

*Proof.* Let C and $C'$ be the chains with m and n elements, respectively. Suppose $m \equiv 0 (mod\ 4)$ and $n \equiv 4 (mod\ 4)$. Suppose $C = a_1 \prec a_2 \prec a_3 \ldots \prec a_m$, $C' = b_1 \prec b_2 \prec b_3 \ldots \prec b_n$ and $L = C]_0^1 C'$. Clearly, L has $m+n$ elements and $m+n$ coverings (edges). Suppose V = $\{a_1, a_2, a_3, \ldots, a_m, b_1, b_2, b_3, \ldots, b_n\}$ and E = {0, 1, 2, … , $m+n$}. Consider a map $\phi : V \to E$ defined as follows :



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$$\phi(a_i)=\begin{cases}\frac{i-1}{2} & \text{if} \quad i=1,3,5,\ldots,m-1\\ m+n-(\frac{i}{2}-1) & \text{if} \quad i=2,4,6,\ldots,\frac{m+n+1}{2}\\ m+n-\frac{i}{2} & \text{if} \quad i=\frac{m+n+1}{2}+2,\frac{m+n+1}{2}+4,\ldots,m\end{cases}$$

$$\phi(b_j)=\begin{cases}\frac{m+n-j}{2} & \text{if} \quad j=1,3,5,\ldots,n\\ \frac{m+n+j-1}{2} & \text{if} \quad j=2,4,6,\ldots,\frac{m+n+1}{2}-2\\ \frac{m+n+j+1}{2} & \text{if} \quad j=\frac{m+n+1}{2},\frac{m+n+1}{2}+2,\ldots,n-1\end{cases}$$

Clearly $\phi$ gives the required graceful labeling for L.

**Theorem 3.6** *Let C and C′ are chains with* $|C|=m\geq 3$ *and* $|C'|=n\geq 1$ *and* $L=C]_0^1C'$. *Then L has graceful labeling if m ≡ 1(mod 4) and n ≡ 3(mod 4).*

*Proof.* Let C and $C'$ be the chains with m and n elements, respectively. Suppose *m ≡ 1(mod 4) and n ≡ 3(mod 4)*. Suppose $C=a_1\prec a_2\prec a_3\ldots\prec a_m$, $C'=b_1\prec b_2\prec b_3\ldots\prec b_n$ and $L=C]_0^1C'$. Here, L has $m+n$ elements and $m+n$ coverings (edges). Suppose V = $\{a_1,a_2,a_3,\ldots,a_m,b_1,b_2,b_3,\ldots,b_n\}$ and E = {0, 1, 2, ... , *m+n*}. Consider a map $\phi:V\to E$ defined as follows :

$$\phi(a_i)=\begin{cases}\frac{i-1}{2} & \text{if} \quad i=1,3,5,\ldots,m\\ m+n-(\frac{i}{2}-1) & \text{if} \quad i=2,4,6,\ldots,\frac{m+n}{2}\\ m+n-\frac{i}{2} & \text{if} \quad i=\frac{m+n}{2}+2,\frac{m+n}{2}+4,\ldots,m-1\end{cases}$$

$$\phi(b_j)=\begin{cases}\frac{m+n+j-1}{2} & \text{if} \quad j=1,3,5,\ldots,\frac{m+n}{2}-1\\ \frac{m+n+j+1}{2} & \text{if} \quad j=\frac{m+n}{2}+1,\frac{m+n}{2}+3,\ldots,n\\ \frac{m+n-j}{2} & \text{if} \quad j=2,4,6,\ldots,n-1\end{cases}$$

Clearly $\varphi$ gives the required graceful labeling for L.

**Theorem 3.7** *Let C and C′ are chains with* $|C|=m\geq 3$ *and* $|C'|=n\geq 1$ *and* $L=C]_0^1C'$. *Then L has graceful labeling if m ≡ 0(mod 4) and n ≡ 0(mod 4).*

*Proof.* Let C and $C'$ be the chains with m and n elements, respectively. Suppose *m = 0(mod 4) and n = 0(mod 4)*. Suppose $C=a_1\prec a_2\prec a_3\ldots\prec a_m$, $C'=b_1\prec b_2\prec b_3\ldots\prec b_n$ and $L=C]_0^1C'$. L has *m+n* elements and *m+n* coverings (edges). Suppose V = $\{a_1,a_2,a_3,\ldots,a_m,b_1,b_2,b_3,\ldots,b_n\}$ and E = {0, 1, 2, ... , *m+n*}. Consider a map $\phi:V\to E$ defined as follows :

$$\phi(a_i)=\begin{cases}\frac{i-1}{2} & \text{if} \quad i=1,3,5,\ldots,m-1\\ m+n-(\frac{i}{2}-1) & \text{if} \quad i=2,4,6,\ldots,\frac{m+n}{2}\\ m+n-\frac{i}{2} & \text{if} \quad i=\frac{m+n}{2}+2,\frac{m+n}{2}+4,\ldots,m\end{cases}$$

$$\phi(b_j)=\begin{cases}\frac{m+n+j-1}{2} & \text{if } j=1,3,\ldots 5,\frac{m+n}{2}-1\\ \frac{m+n+j+1}{2} & \text{if } j=\frac{m+n}{2}+1,\frac{m+n}{2}+3,\ldots,n-1\\ \frac{m+n+j+1}{2} & \text{if } j=2,4,6,\ldots,n\end{cases}$$



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Clearly $\varphi$ gives the required graceful labeling for L.

**Theorem 3.8** *Let C and C′ are chains with* $|C| = m \geq 3$ *and* $|C'| = n \geq 1$ *and* $L = C]_0^1 C'$ *. Then L has graceful labeling if m ≡ 3(mod 4) and n ≡ 0(mod 4).*

*Proof.* Let C and $C'$ be the chains with m and n elements, respectively. Suppose *m ≡ 3(mod 4) and 0  1(mod 4)*. Suppose $C = a_1 \prec a_2 \prec a_3 \ldots \prec a_m$, $C' = b_1 \prec b_2 \prec b_3 \ldots \prec b_n$ and $L = C]_0^1 C'$. Here, L has *m+n* elements and *m+n* coverings (edges). Suppose V = $\{a_1, a_2, a_3, \ldots, a_m, b_1, b_2, b_3, \ldots, b_n\}$ and E = {0, 1, 2, … , *m+n*}. Consider a map $\phi : V \rightarrow E$ defined as follows :

$$\phi(a_i) = \begin{cases} \dfrac{i-1}{2} & \text{if } i = 1,3,5,\ldots,m \\ m+n-(\dfrac{i}{2}-1) & \text{if } i = 2,4,6,\ldots,\dfrac{m+n+1}{2} \\ m+n-\dfrac{i}{2} & \text{if } i = \dfrac{m+n+1}{2}+2, \dfrac{m+n+1}{2}+4,\ldots,m-1 \end{cases}$$

$$\phi(b_j) = \begin{cases} \dfrac{m+n-j}{2} & \text{if } j = 1,3,\ldots 5,m \\ \dfrac{m+n+j-1}{2} & \text{if } j = 2,4,6,\ldots,\dfrac{m+n+1}{2}-2 \\ \dfrac{m+n+j+1}{2} & \text{if } j = \dfrac{m+n+1}{2}, \dfrac{m+n+1}{2}+2,\ldots,n \end{cases}$$

Clearly $\varphi$ gives the required graceful labeling for L.

## Conclusion

In this paper, we introduced graceful labeling for finite posets. We obtained graceful labeling of some finite posets such as a chain, a fence, and a crown. Also, we obtained graceful labeling of an adjunct sum of two chains concerning an adjunct pair (*0,1*). We raise the problem of finding graceful labeling of an adjunct sum of two chains concerning an adjunct pair (*a,b*) in general. Further, the problem may be extended to the class of finite dismantlable lattices/posets also.

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